DEFINABILITY OF SOME CLASSES OF MODULES IN ABSTRACT LOGICS

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ABSTRACT. Working with the language of module theory or extensions of itself, we show that the class of finitely presented R-modules is $\Delta(L_{|R|^+\omega})$-definable and the class of Mittag-Leffler R-modules is $\Delta_1(L_{|R|^+\omega_1})$-definable. For free modules, among other things, we prove that if there is a cardinal κ such that every κ-free module is free, then the class of free R-modules is $L_{\infty\infty}$-, $\Delta(L_{\infty\omega})$-definable. Also if this class is $\Delta(L_{\infty\omega})$-definable, and there is a strongly compact cardinal λ, then there exists a cardinal $\kappa > \lambda$ such that κ-free implies free. Around the class of Baer R-modules, we show that every κ-Baer module M is a Baer module, when κ is a strongly or weakly compact cardinal (for κ weakly compact, M must be of size κ). As a consequence, under $V = L$, the class of Baer modules of size $\leq \kappa$ is $L_{\infty\kappa}$-definable We seize our results around free modules to tackle their model-theoretic power through AECs. We demonstrate that the AEC of free modules has the JEP, AP, and NMM. Moreover, such a class is also fully tame.

1. INTRODUCTION.

The application of first-order model theory to algebra has proved to be quite useful. Nevertheless, it is well-known that first-order logic (in the context of infinitary logics denoted $L_{\omega\omega}$) is not expressive enough. For instance, basic classes of abelian groups, such as free or torsion groups, are not expressible in $L_{\omega\omega}$ using the language of group theory. Of course, this also occurs for the corresponding classes of R-modules with the first-order language of the module theory.

Axiomatizing classes of modules is an important reason to look at abstract logics, which are more expressive than $L_{\omega\omega}$. This paper is devoted to provide axiomatizations of several classes of modules in appropriate abstract logics. Sometimes, we succeed in doing this under ZFC alone, but some classes of modules require additional axioms or large cardinal assumptions. In section 2 we describe succinctly abstract logics and the Δ operator, as well as the corresponding logics $\Delta(L)$ and $\Delta_1(L)$. In Section 3, we tackle the classes of finitely presented and Mittag-Leffler R-modules. We achieve a $\Delta(L_{|R|^+\omega})$-, respectively, a $\Delta_1(L_{|R|^+\omega})$-axiomatization. In Section 4, we illustrate the complexity of expressing that a group is free. Then, we turn to R-modules. Here and in the rest of the paper, we heavily rely on the so-called κ-"free"-modules, where "free" means free, or Baer. The definition of κ-free is based on the notion of κ-dense systems. We first weaken this notion, and then we show that for weakly and strongly compact cardinals, both notions coincide. It is shown that every κ-free module M of any size is free when κ is

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a strongly compact cardinal. We have a corresponding result for weakly compact cardinals, when the cardinality of M is $\leq \kappa$. We show some equivalent assertions around $L_{\infty\infty}$-, $\Delta(L_{\infty\infty})$-, $\Delta(L_{\infty\omega})$-definability and when a κ-free module is free. For example, the class of free R-modules is $\Delta(L_{\infty\omega})$-definable if and only if there is a cardinal $\kappa > \lambda$ such that every κ-free R-module is free, where λ is a strongly compact cardinal.

In Section 5, we examine Baer modules over arbitrary domains. First, we verify that the direct limit M of a κ-directed system of Baer modules is a Baer module, whenever κ is a strongly compact cardinal (or a weakly compact cardinal when $|M| = \kappa$). Then, we deduce that for those classes of large cardinals, a κ-free module M is free (when $|M| = \kappa$ in case κ is only weakly compact). A suitable axiomatization of Baer modules is more complicated because their definitions involve arbitrary torsion modules T. Under $V = L$, we acknowledge the existence of a single test module T_κ. Hence, if κ is a weakly compact cardinal, the class of Baer modules of cardinality $\leq \kappa$ can be axiomatized by an $L_{\infty\kappa}$-sentence.

Finally, in Section 6, we consider AECs. We explore further the results established for free modules, which describe a necessary and sufficient condition for the class **F** of free modules to be definable in an infinitary logic. This fact sets out **F** to be an abstract elementary class. We prove such a class has the Joint Embedding Property and the Amalgamation Property. Also, **F** is $(< \omega)$-tame.

2. Preliminaries

2.1. **Abstract logics.** Working with an abstract logic provides a significant extension to expressibility in order to represent additional mathematical concepts. First of all, we must give an acceptable definition of what we mean by an abstract logic. For readers unfamiliar with the terminology and notation of mathematical logic, we consider languages $\mathscr{L}$ with constant, function, and predicate symbols. An $\mathscr{L}$-structure $\mathfrak{A}$ is a system of the form

$$\mathfrak{A} = \langle A, \bar{c}^{\mathfrak{A}}, \bar{f}^{\mathfrak{A}}, \bar{R}^{\mathfrak{A}} \rangle.$$

Here A is a non empty set (the universe of $\mathfrak{A}$) and $\bar{c}^{\mathfrak{A}}$, $\bar{f}^{\mathfrak{A}}$, $\bar{R}^{\mathfrak{A}}$ represent the interpretations, in A, of the symbols in $\mathscr{L}$.

Every abstract logic L has a syntax and a semantics. That is, for every language $\mathscr{L}$ there is a set of $\mathscr{L}$-sentences $L(\mathscr{L})$ and a satisfaction relation $\models_L$ between $\mathscr{L}$-sentences and $\mathscr{L}$-structures.

Definition 2.1. An abstract logic L is a pair $(L, \models_L)$, such that for every language $\mathscr{L}$ the function L yields a collection of sentences over $\mathscr{L}$: $L(\mathscr{L})$. The L-satisfaction relation $\models_L$ is one between $\mathscr{L}$-structures and sentences $\varphi \in L(\mathscr{L})$. Such a relation must attest to the following criteria.

(1) If $\mathfrak{M} \cong \mathfrak{N}$ and $\mathfrak{M} \models_L \varphi$, then $\mathfrak{N} \models_L \varphi$.
(2) For every $\varphi \in L(\mathscr{L})$, there is some $\mathscr{L}_\varphi \subset \mathscr{L}$ with $\varphi \in L(\mathscr{L})$.
(3) Suppose we have a 1-1 application among languages $f : \mathscr{L} \to \mathscr{L}'$, that preserves arity and types of symbols. I.e., $f(\varsigma)$ is a constant/α-relation/β-function symbol, as long as ς is a constant/α-relation/β-function symbol. Then, for any $\varphi \in L(\mathscr{L})$, and any $f[\mathscr{L}]$-structure $\mathfrak{M}$ we obtain

$$\mathfrak{M} \models_L f(\varphi) \Longleftrightarrow \mathfrak{M}^{f^{-1}} \models_L \varphi.$$

Here, $f(\varphi)$ is obtained by replacing every non-logical symbol ς appearing in φ with its image $f(\varsigma)$. And, $\mathfrak{M}^{f^{-1}}$ is the $\mathscr{L}$-structure product of interpreting the symbol ς exactly as $\mathfrak{M}$ renders the symbol $f(\varsigma)$.

Notice that this definition for an abstract logic only pinpoints the dynamics between the models and the formulas. However, it does not clarify the syntax of the formulas, nor does it specify how to compose formulas. We are only considering abstract logics that are called *regular*, (Definition

1.3, [5]). Regular logics are well-behaved with respect to syntax. Meaning that a regular logic has the closure property we would expect for a logic: conjunction, disjunction, negation, existential quantification, atomic sentences, and relativization. For a better understanding of what regular logics are, it is necessary to establish a relationship between abstract logics. Suppose $\mathscr{L}$ is a given language and L is an abstract logic. If $\varphi \in L(\mathscr{L})$, we define its class of models as $\mathrm{Mod}_L(\varphi) = \{\mathfrak{A} : \mathfrak{A} \models_L \varphi\}$. If L_1 and L_2 are two abstract logics, we say that $L_1 \leq L_2$ if for any $\varphi \in L_1(\mathscr{L})$ there is a $\psi \in L_2(\mathscr{L})$ such that $\mathrm{Mod}_{L_1}(\varphi) = \mathrm{Mod}_{L_2}(\psi)$. That is, anything expressed in L_1 can be expressed in L_2. In simpler words, a regular logic L is at least as expressive as first-order logic, i.e., $L_{\omega\omega} \leq L$.

The most involved property for a regular logic is the relativization property. We illustrate this property because it will be central for our purposes. We say that an abstract logic L has the relativization property if for any $\varphi \in L(\mathscr{L} \cup \mathscr{K})$, any constant symbol c not appearing in $\mathscr{L} \cup \mathscr{K}$, $\varphi \in L(\mathscr{L})$, $\theta \in L(\mathscr{K} \cup \{c\})$, there is a $\psi \in L(\mathscr{L} \cup \mathscr{K})$ such that

$$\mathfrak{B} \in \mathrm{Mod}_L(\psi) \iff \theta(\mathfrak{B}) \restriction \mathscr{L} \models_L \varphi.$$

Here $\theta(\mathfrak{B})$ is the collection of $b \in B$ such that $(\mathfrak{B}, b) \models_L \theta(c)$ when c is interpreted as b, and $\theta(\mathfrak{B}) \restriction \mathscr{L}$ means that we forget about any non-logical not appearing in $\mathscr{L}$. Finally, for this to make sense, we must ensure that $\theta(\mathfrak{B})$ is $\mathscr{L}$-closed, i.e., if for any function symbol $f \in \mathscr{L}$, and any $\vec{b} \in \theta(\mathfrak{B})$, it occurs that $f(\vec{b}) \in \theta(\mathfrak{B})$. So, any constant symbol $c_0 \in \mathscr{L}$ must satisfy θ in $\mathfrak{B}$. This ensures that $\theta(\mathfrak{B})$ is an $\mathscr{L}$-structure. In words, we can code satisfaction to a restricted domain of our given model through a precise sentence, as long as everything is well-defined. This sentence is usually denoted as φ^θ, and is called the relativization of φ to θ.

We will deal mainly with the infinitary logics $L_{\kappa\lambda}$, $L_{\infty\omega}$, and $L_{\infty\infty}$. Given cardinals κ and λ, the infinitary logic $L_{\kappa\lambda}$ extends $L_{\omega\omega}$ allowing the conjunction and disjunction of fewer than κ formulas, as long as the formulas gather fewer than $< \lambda$ free variables. $L_{\kappa\lambda}$ also allows quantification over fewer than λ variables. $L_{\infty\omega}$ and $L_{\infty\infty}$ are those infinitary logics such that $\varphi \in L_{\infty\omega}$ if and only if there is a cardinal μ such that $\varphi \in L_{\mu\omega}$ and, similarly $\psi \in L_{\infty\infty}$ if and only if there are cardinals μ, ν in such a way that $\psi \in L_{\mu\nu}$.

We have stated that the main logical device for our endeavors is the infinitary machinery, yet we introduced logic in the abstract. The reason to do so is to use an important logical operator that will extend our infinitary scenario, i.e., the Δ operator.

2.2. **The Δ operator.** Throughout this section, $\mathscr{L}$ will be a language and L an abstract logic. Given $\varphi \in L(\mathscr{L})$, the class of $\mathscr{L}$-structures $\mathrm{Mod}(\varphi)$ is called an $L(\mathscr{L})$-elementary class. Also, If $\mathscr{L}^+$ is a language that extends $\mathscr{L}$ ($\mathscr{L}^+ \supseteq \mathscr{L}$) and $\mathfrak{M}$ is an $\mathscr{L}^+$-structure, we define the reduct of $\mathfrak{M}$ to $\mathscr{L}$ as $\mathfrak{M} \restriction \mathscr{L} = \langle M, P^{\mathfrak{M}}, \dots \rangle$, for every $P \in \mathscr{L}$.

Definition 2.2. A class K of $\mathscr{L}$-structures is Σ in L when there is an $\mathscr{L}^+ \supseteq \mathscr{L}$ and a sentence $\varphi \in L(\mathscr{L}^+)$ in such a way that

$$K = \underbrace{\{\mathfrak{M} \restriction \mathscr{L} : \mathfrak{M} \models \varphi\}}_{=\mathrm{Mod}(\varphi)\restriction\mathscr{L}}. \tag{2.1}$$

In that case, we say φ is Σ over L or φ is a Σ-*definition* of K over L.

Σ-classes are commonly named projective classes. To avoid confussion between the algebraic notion of projective modules and the model-theoretic notion of projective classes, we decided to denote the latter as Σ-classes. The sentence φ defining a Σ class K over L is sometimes referred to as an *implicit definition*. It is implicit in the sense that we consider extra symbols to define K, but we disregard such symbols when dealing with K itself by considering the respective restrictions $\mathfrak{M} \restriction \mathscr{L}$. Notice that any elementary class can be viewed as a Σ class by considering $\mathscr{L}^+ = \mathscr{L}$.

If we identify each Σ class with its defining sentences, we end up dealing with an extension of the logic L. The easiest way to represent a Σ-definition is by extending the use of the existential quantifier, making explicit how we obtained the formula. That is, to define a Σ class, we need a larger language $\mathscr{L}^+$. Thus, the projective sentence may be presented as $\exists \mathscr{L}^+ \varphi$. Since, in general, we do not have at our disposal a quantifier-elimination Theorem, we cannot ensure that the complement of a projective class is again projective. This is why the extension obtained by considering every projective definition is not a regular logic; the negation clause might fail. Hence, one might be interested in Δ-classes.

Definition 2.3. A class K of $\mathscr{L}$-structures is Δ over L if K and its complementary class have a Σ-definition over L. These sentences are called Δ over L.

The collection of Δ-sentences over L, denoted as $\Delta(L)$, is a new logic that does satisfy the negation property, and thus, it is regular. The logic $\Delta(L)$ is known as the Δ-closure of L.

Theorem 2.4. *$\Delta(L)$ is regular, whenever L is regular.*

Proof. Since we could not find a suitable source containing a proof of this, we include it here. To verify the conjunction and disjunction property, take $\varphi_i \in L(\mathscr{L}_i)$. Let us fix $\mathscr{L}^+ = \mathscr{L}_0 \cup \mathscr{L}_1$. Clearly $\varphi_0 \vee \varphi_1, \varphi_0 \wedge \varphi_1 \in L(\mathscr{L}^+)$. To ensure the existential property, it is enough to note that the very same definition of an existential formula requires us to consider an expansion through a constant symbol. To verify the relativization property, we consider a language $\mathscr{L}$, expansions $\mathscr{L}_0, \mathscr{L}_1$, and sentences $\theta(c) \in L(\mathscr{L}_0 \cup \{c\})$, $\varphi \in L(\mathscr{L}_1)$. The hypothesis ensures that there is a $\psi \in L(\mathscr{L}_0 \cup \mathscr{L}_1)$ such that

$$\mathfrak{M} \models \psi \iff \theta(\mathfrak{M}) \upharpoonright \mathscr{L}_1 \models \varphi.$$

Here $\theta(\mathfrak{M}) \upharpoonright \mathscr{L}_1$ defines the $\mathscr{L}_1$-model whose universe is the set of θ-witnesses $a \in M$. Thus, $\psi \in L(\mathscr{L}_0 \cup \mathscr{L}_1)$. Note that up to this point, we have not used the full expressive power of $\Delta(L)$; so this argument also holds for the class of projective sentences. Finally, the negation property follows from the sheer definition of a Δ sentence over L. □

An abstract logic that satisfies $\Delta(L) \equiv L$ is said to have the *Δ-interpolation property*. These logics have a balance between syntax and semantics. The operator Δ is thought of as a closeness operator, and $\Delta(L)$ as an "approximation" to the second-order version of L, but within L itself. This is made precise through the following theorem.

Theorem 2.5. *The Δ operator has the following properties:*

(1) $L \leq \Delta(L)$,
(2) $\Delta(\Delta(L)) \equiv \Delta(L)$, and
(3) for $L \leq L'$ we have $\Delta(L) \leq \Delta(L')$.

Proof. [4, Theorem 7.2.4(ii), p. 71]. □

So, Δ truly constitutes a closure operator. Next, we will define a stronger operator than Δ, which is called Δ_1 to emphasize the distinction. We first define a strengthening of a projective class.

Definition 2.6. A class of $\mathscr{L}$-structures K is a *Σ_1-class over L*, or $\Sigma_1(L)$ for short, whenever there is a language $\mathscr{L}^+ \cup \{c\} \supseteq \mathscr{L}$, an $\mathscr{L}^+ \cup \{c\}$-sentence $\theta(c)$ and an $\mathscr{L}^+$-sentence φ such that $\mathfrak{N} \in K$ exactly when we can find an $\mathscr{L}^+$-extension $\mathfrak{N}^+$ such that $\mathfrak{N} = \theta(\mathfrak{N}^+) \upharpoonright \mathscr{L}$ and $\mathfrak{N}^+ \models \varphi$. Here $\theta(\mathfrak{N}^+) = \{a \in N^+ : (\mathfrak{N}^+, a) \models \theta(c)\}$.

What separates $\Sigma(L)$-classes and $\Sigma_1(L)$-classes is that Σ only enriches the language, without enlarging the domain of the structure. In contrast, a Σ_1-class does enlarge the domain of the structures. Also, it is worth noting that the constant c is only introduced as a way to use formulas, logic $L_{\mu\lambda}$. In order to achieve an $L_{\mu\lambda}$-elementary definition, we would need results analogous to those from [29]. This does not seem to be as immediate because the authors use an arrow relation for colored trees. We would need a strengthening of such an arrow relation, one that we know nothing of. That is, we do not know whether the strengthening is a ZFC-theorem or not. Even less, it is not known if this is a ZFC-independent statement.

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