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A gap 1 cardinal transfer theorem

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We extend the gap 1 cardinal transfer theorem $(\kappa^+, \kappa) \longrightarrow (\lambda^+, \lambda)$ to any language of cardinality $\leq \lambda$, where λ is a regular cardinal. This transfer theorem has been proved by Chang under GCH for countable languages and by Silver in some cases for bigger languages (also under GCH). We assume the existence of a coarse $(\lambda, 1)$-morass instead of GCH.

1 Introduction

Morasses have been developed by R. Jensen in order to solve some strong combinatorial problems, like the gap n transfer theorem of model theory. The proof that a full morass exists in L heavily relies on the fine structure of L.

In [3] a full morass is described through seven axioms. If we only consider the first five of them, we get a so called *coarse morass*. The proof of that a full morass exists in L is, of course, a proof of the existence in L of a coarse morass. But a coarse morass exists in models of ZFE under weaker hypothesis than $V = L$.

What seems to be more important is that with a coarse morass we are able to prove several combinatorial results (see [4]) and carry out complicated constructions like the one described in this paper. Coarse morasses share some of the most important features of full morasses but are easier to use. We shall describe in this paper the construction of a general structure of cardinality λ^+ with a unary relation of cardinality λ, for λ a regular cardinal. The framework we develop is expected to handle other problems not necessarily of the same nature.

For instance, let κ be a cardinal. An R-module M over a ring R is called *κ-torsionless* if the cardinality of R is less than κ and every submodule N of M of cardinality less than κ is torsionless (it can be embedded in a product of copies of R). We can prove (see [10]) that every κ-torsionless R-module M of cardinality κ, for κ a weakly compact cardinal, is torsionless. A fundamental question is: does there exist a κ-torsionless R-module which is not torsionless ($R \neq \mathbb{Z}$)? We are able to answer positively this question (see [10]) by using a coarse morass and a combinatorial principle valid for non compact cardinals, following the construction described in this paper.

We feel that coarse morasses in combination with the square principle are the most likely structures to find counterexamples of some cases of the selection theorem (see [11, Theorem 1.3, Corollary 1.4, p. 286]) in E-recursion for singular cardinals.

We believe that a coarse morass and the construction described here provide tools which are potentially valuable to anyone who pretends to construct structures with particular subsets.

Now we describe the problem in this paper. Let $\mathfrak{L}$ be a language with at least one unary predicate symbol U. We say that an $\mathfrak{L}$-structure $\mathfrak{A}$ *has type* (κ, μ) if $\mathfrak{A} = \langle A, U^{\mathfrak{A}}, \dots \rangle$ and $|A| = \kappa$, $|U^{\mathfrak{A}}| = \mu$. An $\mathfrak{L}$-theory T admits (κ, μ) if it has a model $\mathfrak{A}$ of type (κ, μ). We write

$$(\kappa, \mu) \Rrightarrow (\kappa', \mu')$$

if whenever T admits (κ, μ) witnessed by a model $\mathfrak{A}$, T also admits (κ', μ') witnessed by a model $\mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$.

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[5] R. B. Jensen and C. Karp, Primitive recursive set functions. In: Axiomatic Set Theory (D. Scott, ed.). Proceedings of Symposia in Pure Mathematics **13**, pp. 143 – 167 (AMS, 1971).
[6] R. B. Jensen, The fine structure of the constructible hierarchy. Ann. Math. Logic **4**, 229 – 308 (1972).
[7] R. B. Jensen, private communication.
[8] R. B. Jensen, The (κ, β)-morass. Unpublished manuscript.
[9] J. Kennedy and S. Shelah, On regular reduced products. J. Symbolic Logic **67**, 1169 – 1177 (2002).
[10] P. M. Iturralde, J. A. N. Valencia, and L. M. Villegas-Silva, Weakly compact cardinals and κ-torsionless modules. In preparation.
[11] G. E. Sacks, Higher Recursion Theory (Springer-Verlag, 1990).
[12] L. Stanley, A short course on gap-one morasses with a review of the fine structure of L. In: Surveys in Set Theory, pp. 197 – 243. London Math. Soc. Lecture Note Series 87 (Cambridge University Press, 1983).
[13] L. M. Villegas-Silva, The two cardinal transfer theorem for languages of arbitrarily cardinality. Submitted to the Journal of Symbolic Logic.
[14] H. J. Wunderlich, Über zwei-Kardinalzahl-Probleme. Diplomarbeit, Universität Freiburg, 1981.

