THE GAP-TWO CARDINAL PROBLEM. THE SINGULAR CASE
PART I: THE SQUARE MORASS

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ABSTRACT. In this paper we construct a so called Square Morass under $V = L$. That is, a Morass that involves square sequences in each of its levels with some preservation properties between them. This Morass will be used in part 2 to prove the gap-2 cardinal transfer theorem, the singular case, assuming the axiom of constructibility.

1. INTRODUCTION

As it is mentioned in the abstract, we will set up a square $(\kappa, 1)$-Morass, where κ is a regular uncountable cardinal. The Morass shall comprise a square sequence in each of its level. As we will see, this figure is indispensable to solve the Gap 2 Cardinal transfer Theorem for the singular case.

The two cardinal problem is described next. Let $\mathscr{L}$ be a first order language with at least an unary predicate symbol U. An $\mathscr{L}$-structure $\mathfrak{A}$ is said to be of type (ξ, ζ), when $|A| = \xi$ and $|U^{\mathfrak{A}}| = \zeta$. Given the cardinals $\lambda < \kappa$ and $\eta < \mu$, we write

$$(\kappa, \lambda) \twoheadrightarrow (\mu, \eta)$$

to express that given an $\mathscr{L}$-structure $\mathfrak{A}$ of type (κ, λ), we can find an $\mathscr{L}$-structure $\mathfrak{B}$ of type (μ, η) sucht that $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$.

The specific case

$$(\kappa^{++}, \kappa) \twoheadrightarrow (\lambda^{++}, \lambda)$$

is known as the Gap 2 Cardinal Transfer Theorem. If λ is singular, we are dealing with the singular case.

The Gap 1 problem regular has been solved under distinct hypothesis. Among others, $V = L$, GCH, or the existence of a rough Morass ([V18b]). The singular case was solved by R. Jensen under GCH (or even a weaker assumption, namely $2^{\underset{\smile}{\lambda}} = \lambda$) and the existence of a $\square_\lambda$-sequence (for a presentation of these results: under GCH see the Appendix by J. Silver in[Jen72]; under $2^{\underset{\smile}{\lambda}} = \lambda$ see ([Jen3])).

The Gap 2 problem regular was solved by Jensen for countable languages (see for example [Dev84]) using a $(\lambda^+, 1)$-morass. One can guarantee the existence of such a Morass under $V = L$. In [Vi18] it is shown that the Gap 2 regular problem is also true in $V = L$ for uncountable languages.

The story of the Gap 2 problem singular deserves a more detailed description. It is known that R. Jensen proved that the theorem is true under $V = L$. He also used a $(\lambda^+, 1)$-morass and a very involved model theory. Nevertheless, the proof has been never published and the manuscript is lost. Several years ago I started to learn about morasses and R. Jensen suggested me to provide a complete and detailed proof of the problem. Since then, I has been trying to fulfill the task. At the beginning, the plan was to prove the theorem for first order logic. Thereafter, I started to work with infinitary logics. In specific, with consistency properties and end-extensions. We

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provided solutions for the Gap 1 problem regular ([AgHeVia]) and singular ([AgHeVib]) for infinitary logics in the following sense. Let $\mathscr{L}$ be a first order language. Consider the infinitary logic $L_{\kappa\omega}(\mathscr{L})$, where κ is an uncountable cardinal. A fragment $\mathbb{A}$ of this logic is a nicely closed set of $L_{\kappa\omega}(\mathscr{L})$-formulas containing the first order fragment $L_{\omega\omega}(\mathscr{L})$. We work with p.r. closed fragments $\mathbb{A}$ of size κ in the regular case, or size $< \kappa$ in the singular case.

The Gap n Cardinal Transfer problem is

$$(\kappa^{(n)}, \kappa) \twoheadrightarrow_{\mathbb{A}} (\lambda^{(n)}, \lambda),$$

which is to be understood like the first order case but the $\mathscr{L}$-structure of type $(\lambda^{(n)}, \lambda)$ shall be elementary $\mathbb{A}$-equivalent to $\mathfrak{A}$. We also solved the Gap 2 problem regular with omitting types for a fragment $\mathbb{A}$ ([ViVi]). All these results embrace the first order case, of course.

So, it was natural to look for a solution of the Gap 2 problem singular in the realm of infinitary logic. We will present a complete solution of the singular case for a fragment $\mathbb{A}$ of size $< \lambda$. It is important to notice that an infinitary satisfiable theory can lack of arbitrary large models. Thus, the Gap 2 problem singular will be solved under the hypothesis $\lambda < \kappa$. But, when $\mathbb{A}$ is the first order fragment or the $\mathbb{A}$-theory does have arbitrary large models, we can handle also the case $\lambda > \kappa$.

The proof of the problem is rather lengthy. It consists in three parts. The first part provides the square Morass required for the solution. The second takes care of the model theory, producing end $\mathbb{A}$-elementary extensions. The third part construct the model $\mathfrak{B}$ through end $\mathbb{A}$-elementary extensions using the square morass. It has been claimed that these processes can be accomplished with a normal Morass. That is, not necessarily a square morass. As the reader will see in part two, it is impossible to succeed when the morass does not contains square sequences in its levels, even for the first-order case.

The construction of the square morass heavily relies on the Σ^*-fine structure ([Jen]). Part of this development takes place in arbitrary acceptable structures J^A_α. That is, several results, for instance, the lift up, hold in general acceptable structures. We appeal to a so called Smooth Category, whose properties are established also in the general case. However, the square morass is constructed in L.

I would like to thank Ronald Jensen for suggesting me this research. He also helped me through the years I was completing it. I greatly appreciate his generosity of time and energy; always been available to answer my uncountable many questions about morasses and fine structure.

I have decided to present a complete construction of the morass. First we introduce the Σ^*-fine structure, only the primary notions and results. Then, we establish some necessary results in general form. We provide a fairly complete construction of a liftup for acceptable structures. We follow [Jen]. We have acceptable structures $M = \langle J^A_\gamma, B\rangle$, a cardinal κ according to M such that J^A_κ thinks that there exists a bigger cardinal. This has consequences in the liftup.

After that, we recall the notion of Smooth Category ([JeZe00]). This category produces square sequences, which together with a suitable definition of the morass order allow us to obtain the square Morass.

2. Σ^*-FINE STRUCTURE

As we already mentioned, we rely in the Σ^*-fine structure. In this section we collate definitions and results required in the rest of the paper. Any undefined notion can be fuound in [Jen].

Definition 2.1. The J_α hierarchy is defined by recursion as follows.

$$\begin{aligned} J_\omega &= Rud(\emptyset) \\ J_{\beta+\omega} &= Rud(J_\beta) \qquad \lim(\beta) \\ J_\lambda &= \bigcup_{\gamma<\lambda} J_\gamma \qquad \text{when } \lambda \text{ is a limit of limits ordinals} \end{aligned}$$

We set $L = \bigcup_\alpha J_\alpha$. Then ([Jen, p.66])

$$\mathbb{P}(J_\alpha) \cap J_{\alpha+\omega} = Def(J_\alpha).$$

Given a class A we can form the constructible hierarchy $\langle J_\alpha^A : \alpha \in On\rangle$ relativized to A. Let $A \subseteq V$. The J_α^A hierarchy is defined by recursion.

$$\begin{aligned} J_\alpha^A &= \langle J_\alpha[A], \in, A \cap J_\alpha[A]\rangle \\ J_\omega[A] &= Rud_A(\emptyset) = H_\omega \\ J_{\beta+\omega}[A] &= Rud_A(J_\beta) \qquad \lim(\beta) \\ J_\lambda[A] &= \bigcup_{\nu<\lambda} J_\nu[A] \qquad \text{for } \lambda \text{ a limit of limits ordinal} \end{aligned}$$

Then, we set

$$\begin{aligned} L[A] = J[A] &= \bigcup_{\alpha\in Or} J_\alpha[A] \\ L^A = J^A &= \langle L[A], \in, A \cap L[A]\rangle \end{aligned}$$

The new notion of acceptability.

Definition 2.2. We say that J_α^A is acceptable if and only if for all $\beta \leq \nu < \alpha$ limit ordinals occurs:

(a) If $a \subseteq \beta$ and $a \in J_{\nu+\omega} - J_\nu^A$, then $|\nu| \leq \beta$ in $J_{\nu+\omega}^A$.
(b) If $x \in J_\beta^A$ and φ is Σ_1-formula such that $J_{\nu+\omega} \models \varphi[\beta, x]$ but $J_\nu^A \not\models \varphi[\beta, x]$, then $|\nu| \leq \beta$ in $J_{\nu+\omega}^A$.

A J-model $\langle J_\alpha^A, \vec{B}\rangle$ is acceptable if and only if J_α^A is acceptable.

Lemma 2.3. *Let $M = \langle J_\alpha^A, B\rangle$ be acceptable and let $\gamma > \omega$ be a cardinal in M. Then*

$$J_\gamma^A = H_\gamma^A = \bigcup\{u : u \in M : u \textit{ is transitive } \wedge |u|^M < \gamma\}.$$

Proof. [Jen, Corollary 2.5.3, p. 87]. ❐

Definition 2.4. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. We define sets $M^n_{x^{n-1},\ldots,x^0}$ and predicates $T^n(x^n, \ldots, x^0)$ as follows:

$$\begin{aligned} M^0 = M \qquad T^0 = B \qquad M^n_{\vec{x}} = 0 \text{ for } n = 0 \\ M^{n+1}_{\vec{x}} = \langle J^A_{\rho^{n+1}}, T^{n+1}_{\vec{x}}\rangle \qquad \vec{x} = x^n, \ldots, x^0 \\ T^{n+1}(x^{n+1}, \vec{x}) \Leftrightarrow \exists z^{n+1} \exists i\omega (x^{n+1} = (i, z^{n+1}) \wedge M^n_{x^{n-1},\ldots,x^0} \models \varphi_i[z^{n+1}, x^n]). \end{aligned}$$

where $\langle \varphi_i : i < \omega\rangle$ is a canonical enumeration of Σ_1-formulas.

Then

$$T^{n+1}((i, x^{n+1}, x^n, \ldots, x^0) \Leftrightarrow M_{x^{n-1},\ldots,x^0} \models \varphi_i[x^{n+1}, x^n].$$

Definition 2.5. Let $M = \langle J^A_\alpha, B\rangle$ be acceptable. We define the good parameters

$$\begin{aligned}P^0_M &= [On]^{<\omega}\\ P^{n+1}_M &= \{a \in P^n_M : \text{ there is } D \text{ which is } \Sigma^{(n)}_1(M) \text{ in } a \text{ with } D \cap H^{n+1}_M \notin M\}\\ R^0_M &= P^0_M\\ R^{n+1}_M &= \{a \in R^n_M : M^{n,a} = h_{M^{n,a}}(\rho^{n+1} \cup (a \cap \rho^n))\}.\end{aligned}$$

Definition 2.6. π es a $\Sigma^{(n)}_h$ preserving map of $\overline{M}$ to M, in symbols $\pi : \overline{M} \xrightarrow[\Sigma^{(n)}_h]{} M$, if and only if the following hold:

(1) $\overline{M}, M$ are acceptable structures of the same type.
(2) $\pi[H^i_{\overline{M}}] \subseteq H^i_M$ for $i \leq n$.
(3) Set $\varphi \equiv \varphi(v^{j_1}_1, \ldots, v^{j_m}_m)$ be a $\Sigma^{(n)}_h$ formula with a good sequence $\vec{v}$ of variables such that $j_1, \ldots, j_m \leq n$. Let $x_i \in H^{j_i}_{\overline{M}}$ for $i = 1, \ldots, m$. Then:

$$\overline{M} \models \varphi[\vec{x}] \qquad \Leftrightarrow \qquad M \models \varphi[\pi(\vec{x})].$$

If π is $\Sigma^{(n)}_h$-preserving, it is $\Sigma^{(m)}_1$-preserving for $m < n$ and $\Sigma^{(n)}_i$-preserving for $i < h$. If $h \geq 1$, then $\pi^{-1}[H^n_M] \subseteq H^n_{\overline{M}}$. We say that π is strictly $\Sigma^{(n)}_h$ preserving (in symbols $\pi : \overline{M} \xrightarrow[\Sigma^{(n)}_h]{} M$ strictly) if and only if it is $\Sigma^{(n)}_h$ preserving and $\pi^{-1}[H^n_M] \subseteq H^n_{\overline{M}}$. Only if $h = 0$ can the embedding fail to be strict.

Lemma 2.7. *The condensation lema for the J-hierarchy is as follows. Let $M = J_\beta$, let $\pi : J_\delta \xrightarrow[\Sigma^{(n)}_1]{} J_\beta$. Assume $\rho^{n+1}_M \leq \kappa < \rho^n_M$, with $\pi(\overline{\kappa}) = \kappa$ and $\pi(\overline{p}) = p_{J_\beta} - \kappa$. Then $\overline{p} = p_{J_\delta} - \overline{\kappa}$.*

Proof. Apart from the notation change (we use ρ^n instead of $\omega\rho^n$), this is [We10, Lemma 1.22, p. 670]. ❒

Let $a \in [On_M]^{<\omega}$. We set $a^{(i)} = a \cap \rho^i$ for $i < \omega$.

Definition 2.8. Let $a \in [On_M]^{<\omega}$. we define partial maps h_a with domain $\omega \times H^n_M$ to H^n_M by:

$$h^i_a(i,x) \simeq h_{M^{n,a}}(i,(x,a^{(n)}).$$

Then h^n_a is uniformly $\Sigma^{(n)}_1$ in $a^{(n)}, \ldots, a^{(0)}$. We then define maps $\tilde{h}^n_a$ from $\omega \times H^n_M$ to $H^0_M = M$ by:

$$\begin{aligned}\tilde{h}^i_a &\simeq h^0_a(i,x)\\ \tilde{h}^{n+1}_a(i,x) &\simeq \tilde{h}^n_a((i)_0, h^{n+1}_a((i)_1, X)).\end{aligned}$$

Then $\tilde{h}^n_a$ is a good $\Sigma^{(n)}_1$ function uniformly in $a^{(n)}, \ldots, a^{(0)}$. It is clear that, if $a \in R^{n+1}_M$, then

$$h^{(n)}_a[\omega \times \rho^{n+1}] = H^n_M.$$

Therefore, if $a \in R^{n+1}_M$, then $\tilde{h}^n_a[\omega \times \rho^{n+1}] = M$.

Lemma 2.9. *Let $\overline{M}, M$ be acceptable and let $\sigma : \overline{M} \xrightarrow[\Sigma^{(m)}_1]{} M$. Suppose that there exists $p \in P^m_M$ with $p \in ran(\sigma)$. Then σ is $\Sigma^{(m+1)}_1$ preserving.*

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