# THE HANF NUMBER AND THE TWO-CARDINAL THEOREM FOR INFINITARY LOGIC

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Abstract. We show a generalization of the Morley's two cardinal Theorem in the infinitary logic $L_{\kappa\omega}$, where $\kappa$ is a regular cardinal and we consider pr closed fragments. There are a number of related results concerning Hanf number.

## 1. Introduction

While we were working on an application of the (infinitary) Morley's two-cardinal transfer theorem to homogeneous models, we came across a Theorem of Barwise [Ba75, Theorem 4.3, p. 277]. It provides a more general setting than [Ke71, Theorem 23]. In order to describe this issue we need some informal definitions. Let $\mathcal{L}$ be a first order language and assume that we code the formulas of $L_{\kappa\omega}(\mathcal{L})$ as sets in $H_{\kappa^+}$. Suppose that $\mathbb{A}$ is a transitive set. Then, $\mathcal{F} = \mathbb{A} \cap H_{\kappa^+}$ is a fragment of $L_{\kappa\omega}(\mathcal{L})$ when $\mathbb{A}$ satisfies suitable closure properties. In particular, this happens when $\mathbb{A}$ is an admissible set or a primitive recursive (pr) closed set. Here is important to realize that any admissible set is pr closed, but there are non-admissible pr closed sets. For instance, if $\alpha$ is an admissible ordinal, the first pr closed ordinal above $\alpha$ is not admissible. Let $T$ be a theory on $\mathcal{F}$ and suppose that $\mathcal{L}$ contains at least a unary predicate symbol $P$. A model $\mathfrak{A}$ of $T$ is of type $(\kappa, \lambda)$ if $\kappa$ and $\lambda$ are infinite cardinals with $\lambda \leq \kappa$, and $|A| = \kappa$ and $|P^{\mathfrak{A}}| = \lambda$. The aforementioned Morley Theorem is stated next.

**Theorem 1.1** (Morley). *([Ke71, Theorem 23]) Let $\mathcal{F}$ be a countable fragment of $L_{\omega_1\omega}$ and suppose that for any $\alpha < \omega_1$ there exists a cardinal $\kappa \geq \omega$ such that $T$ has a model of type $(\beth_\alpha(\kappa), \kappa)$. Then, for all cardinals $\lambda \geq \omega$, $T$ has a model of type $(\lambda, \omega)$.*

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In [Ba75, Theorem 4.3, p. 277] it is proved next Theorem, where the Hanf number depends on certain ordinal $h(\mathcal{F})$ for admissible fragments $\mathcal{F}$ of $L_{\kappa^+\omega}$.

**Theorem 1.2.** *Let $\mathcal{F}$ be an admissible fragment of size $\kappa$ of some $L_{\mu\omega}(\mathcal{L})$ logic. Assume that $\alpha = h(\mathcal{F})$, $T$ is a $\Sigma_1$-theory of $\mathcal{F}$ and that for any $\beta < \alpha$ there is a $\lambda \geq \kappa$ such that $T$ admits $(\beth_\beta(\lambda), \lambda)$. Then, $T$ admits $(\eta, \kappa)$ for every cardinal $\eta \geq \kappa$.*

The Hanf number of $L_{\omega_1\omega}$ is $\beth_{\omega_1}$, a fact used to prove the countable version of Morley's Theorem. The proof of Theorem 1.2 is based on a relationship between $h(\mathcal{F})$ and the Hanf number of $L_{\kappa\omega}$, when $\mathcal{F}$ is an admissible fragment. The disadvantage of Theorem 1.2 is the use of an admissible fragment, and that it entails a $\Sigma_1$ theory in the fragment. It is not common for the theory $T$ to be $\Sigma_1$. Such requisite is often absent or at least difficult to verify. To elude this obstacle, we look for a version of Theorem 1.2 for pr closed fragments $\mathcal{F}$ and also free from the $\Sigma_1$ restriction on $T$. In this case, an ordinal $\mathfrak{a}(\mathcal{F})$ appears (replacing $h(\mathcal{F})$), which, in general, does not allow a simple representation as in $L_{\omega_1\omega}$. We shall dedicate some effort to characterize this ordinal for pr closed fragments. For the fragment $\mathcal{F}$ in Theorem 1.2 we use, instead, $\mathfrak{a}(\mathcal{F})$. Aiming to this, we turned to the outstanding work of M. Dickmann [Di75, chapter IV]. Using frames [Di75, Definition 4.5.1] (to some extent a weaker notion than fragment) he works out results of Kunen and Barwise. Dickmann provides a relationship between Hanf numbers and $\mathfrak{a}(\kappa^+)$ for frames $\mathcal{F}$; so, if $\mathfrak{h}_\kappa$ is the Hanf number of $L_{\kappa^+\omega}$, then

$$\mathfrak{h}_\kappa = \beth_{\mathfrak{a}(\kappa^+)}.$$

By definition, $\mathfrak{h}_\kappa$ is computed using sets of $L_{\kappa^+\omega}(\mathcal{L})$-sentences for languages $\mathcal{L}$ of size at most $\kappa$. There is absolutely no harm to enlarge frames to pr closed fragments containing such sets of sentences. So, we want to apply this computation to the proof of the Theorem 1.2 for pr closed fragments. Nevertheless, there is a fundamental flaw in a proof of Dickmann around this issue. Indeed, he introduces a definition ([Di75, Definition 4.5.2]) of consistency properties $\Gamma$ for $L_{\kappa\omega}$; among other properties, (item (iii)) if $\Sigma \in \Gamma$, and $\Psi$ is an infinitary conjunction, then $\Sigma \cup \{\psi\} \in \Gamma$ for every conjunct $\psi$ in $\Psi$. Then, he defines [Di75, Definition 4.5.3] a closed set $\Sigma$ of $\mathcal{F}$-sentences ($\mathcal{F}$ a frame) as a set with several closure characteristics. For instance, if $\bigwedge \Psi \in \Sigma$, then $\Psi \subseteq \Sigma$. This means that $\Sigma$ can be uncountable. The next step is to find a model for any closed set, and he refers to the corresponding proof in [Ke71, Lecture3]. The trouble is that Keisler treats only countable set of $\mathcal{F}$-sentences, which can be closed in countably many steps (he uses countable languages $\mathcal{L}$ and $L_{\omega_1\omega}(\mathcal{L})$). Thus, Keisler takes an element $s$ of his consistency property $S$, and closes it by recursion on $\omega$ by applying the rules he has available. In case of $L_{\kappa\omega}$, the recursion occurs on the size of the fragment $\mathcal{F}$, that is, in at least $\kappa > \omega$ steps. The consistency property of Dickmann gives no indication of how to take care of limit steps. The Keisler's proof does not work when there are more that $\omega$ elements in the set to be closed. This issue is not a minor detail. We provide a correct proof for several Dickmann's Theorems and Lemmata around this matter. We turn to a consistency property of Karp-Green ([Gr72] and [Gr75]), and we succeed by generalizing Theorem 1.2 to pr closed fragments. It seems apparent that there is a wealth of details to take care of. To this end, many outcomes occur using Green-Karp's consistency property. We constitute this as an improvement in infinitary logics.

Once we have this successful outcome, the possibility is open to prove Theorem 1.2 for pr closed fragments. Unfortunately, Barwise proof heavily relies on admissible theory; he uses validity ([Ba75, Definition III.4.1]) and supervalidity properties ([Ba75, Definition VII.2.5]), a complex set of formulas which include proof systems for admissible fragments. So, we further develop suitable tools to achieve a corresponding proof without supervalidity properties. At the end we establish bounds for $\mathfrak{a}(\mathcal{F})$ in our context.

We will state several results of Dickmann [Di75], Barwise [Ba75] and Barwise-Kunen [BaKu71], sometimes without proof (because the given proof in the original source does not depends on admissible fragments or on the Dickmann's consistency property), some results deserve a new proof, and some proofs ask for minor modifications.

There is a sequel for this paper, [GaVaVi], where we exploit several of next results in extending or improving results by Keisler and Dickmann.

## 2. Preliminaries

In this section we collect the Definitions, Theorems and construction we shall use in the rest of the paper. In case we provide a proof, it is sketchy. We use Gothic letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{M}$, ... to denote structures and the corresponding roman letters $A$, $B$, $M$, ... to represent their universes. We assume acquaintance with notions of infinitary logic, as well as their model theory. We refer the reader to [Ke71] or [Di75] for such concepts.

Given cardinals $\kappa$ and $\alpha$, the infinitary logic $L_{\kappa\alpha}$ allows the conjunction and disjunction of fewer than $\kappa$ formulas and quantification over fewer than $\alpha$ variables. If $\mathcal{L}$ is a first order language, $L_{\kappa\lambda}(\mathcal{L})$ represents the infinitary logic based on $\mathcal{L}$. If $C$ is a set of new constants, $\mathcal{L}(C)$ denotes the language obtained by adding the constants in $C$ to $\mathcal{L}$. Then, we build the infinitary logic $L_{\kappa\alpha}(\mathcal{L}(C))$. The set of formulas of $L_{\kappa\alpha}(\mathcal{L})$ will be denoted by $Fml(L_{\kappa\alpha}(\mathcal{L}))$. We will focus on infinitary logics $L_{\kappa\omega}$, where the cardinal $\kappa$ is uncountable and regular. If $\mathfrak{A}$ is an $\mathcal{L}$-structure and $D \subseteq A$, $(\mathfrak{A}, D)$ means that we are working with the extended language $\mathcal{L}(D)$, and for each $d \in D$, the constant symbol $\dot{d}$ is interpreted in $\mathfrak{A}$ as the element $d$. With $Th_{\mathcal{F}}(\mathfrak{M})$ we denote the complete theory of the $\mathcal{L}$-structure $\mathfrak{M}$ in the fragment $\mathcal{F}(M)$.

If $x$ is a set, $TC(x)$ is its transitive closure. For a cardinal $\lambda$, $H_\lambda$ is the collection of sets of hereditary cardinality less than $\lambda$. Whenever we deal with $L_{\kappa\omega}(\mathcal{L})$, unless otherwise stated, we assume that $\mathcal{L}$ is a first order language of regular cardinality less than $\kappa$. For any unexplained notion see [Ke71], [Ba75] or [Di75].

## 3. Primitive recursive functions and fragments

A function $f : V^n \to V$ is called *primitive recursive* (pr) if it is generated by successive applications of the following schemata:

i) $f(x_1, x_2, \ldots, x_n) = x_i$, $1 \leq i \leq n$.

ii) $f(x_1, x_2, \ldots, x_n) = \{x_i, x_j\}$, where $1 \leq i, j \leq n$.

iii) $f(x_1, x_2, \ldots, x_n) = x_i - x_j$, where $1 \leq i, j \leq n$.

iv) $f(x_1, x_2, \ldots, x_n) = h(g_1(x_1, x_2, \ldots, x_n), \ldots, g_k(x_1, x_2, \ldots, x_n))$, where $g_1, \ldots, g_k$, $h$ are pr functions.

v) $f(y, x_1, x_2, \ldots, x_n) = \bigcup_{z \in y} g(z, x_1, x_2, \ldots, x_n)$, where $g$ is a pr function.

vi) $f(y, x_1, x_2, \ldots, x_n) = g(y, x_1, x_2, \ldots, x_n, \langle f(z, x_1, x_2, \ldots, x_n) \mid z \in y \rangle)$, where $g$ is a pr function.

This kind of set functions are studied by Jensen and Karp in [JeKa71]. We call $R \subset V^n$ a pr relation, if its characteristic function is a pr function. For our purposes, it is relevant to note that the function $TC(x)$ (transitive closure of $x$), and the relations $dom(x)$ (domain of $x$) and $ran(x)$, among others, are pr. The set $x$ is pr closed if for any pr function $f : V^n \to V$ and every $a_1, \ldots, a_n \in x$, it occurs that $f(a_1, \ldots, a_n) \in x$. We let $prC(z)$ denote the pr closure of a set $z$. If $z$ is a set, then

$$prC(z) = \{f(u) : u \in z, f \text{ is a pr function}\}.$$

Hence, if $z$ is an infinite set, $|prC(z)| = |z|$.

In order to define our fragment we should supply an adequate codification for $L_{\kappa\omega}(\mathcal{L})$-formulas. The reader is addressed to [Ka68] for details. Going forward, we always assume that $Fml(L_{\kappa\omega}(\mathcal{L}))$ consists of codes of formulas. From now on, we will make no distinction between formulas and their codes. The set $Fml(L_{\kappa\omega}(\mathcal{L}))$ of $L_{\kappa\omega}(\mathcal{L})$-formulas is the smallest set which contains all primitive formulas of $\mathcal{L}$, and it is closed under $\neg, \wedge, \vee, \to, \leftrightarrow, \exists x, \forall x$ (in fact, $L_{\omega\omega}(\mathcal{L}) \subseteq L_{\kappa\omega}(\mathcal{L})$). Furthermore, it satisfies the following properties.

- If $x$ is a variable and $\varphi \in Fml(L_{\kappa\omega}(\mathcal{L}))$, then $\forall x\varphi, \exists x\varphi \in Fml(L_{\kappa\omega}(\mathcal{L}))$.
- If $f = \langle \varphi_i | i < \gamma \rangle$ and $\varphi_i \in Fml(L_{\kappa\omega}(\mathcal{L}))$ for $i < \gamma < \kappa$, then $\bigvee f, \bigwedge f \in Fml(L_{\kappa\omega}(\mathcal{L}))$.

**Definition 3.1.** $\mathcal{F}$ is called a *pr closed fragment* of $L_{\kappa\omega}(\mathcal{L})$, if it contains the set of first order formulas $Fml(\mathcal{L})$ and there exists a non-empty transitive pr closed set $A$ such that $\mathcal{F} = Fml(L_{\kappa\omega}(\mathcal{L})) \cap A$.

In what follows, whenever we state $\mathcal{F}$ is a fragment, we mean $\mathcal{F}$ is a pr closed fragment. Observe that, for any first order language $\mathcal{L}$ and each fragment $\mathcal{F}$, $L_{\omega\omega}(\mathcal{L}) \subseteq \mathcal{F}$ holds. Concerning our fragments, there are some points to be made.

**Proposition 3.2.** *Let $\kappa$ be a regular cardinal. Let $\mathcal{F}$ be a fragment of $L_{\kappa\omega}(\mathcal{L})$ of regular size less than $\kappa$. Then, $\mathcal{F} \in H_\kappa$. In particular, if $\mathcal{F}$ is a fragment of $L_{\kappa\omega}$, such that $|\mathcal{L}| \leq |\mathcal{F}| = \lambda < \kappa$, then $\mathcal{F} \in H_\kappa$.*

*Proof.* We proceed by induction on the complexity of the formulas to prove that if $\varphi \in \mathcal{F}$, then $\varphi \in H_\kappa$, so $\mathcal{F} \subseteq H_\kappa$. Hence, $\mathcal{F} \in H_\kappa$ since $|\mathcal{F}| < \kappa$. □

*Proof.* Let $\mu$ be a cardinal with $\mu \geq \kappa$. We suppose again that $\mathcal{F}$ has Skolem functions. According to Theorem 6.19, $T$ has a model $\mathfrak{M} = \langle M, U, \dots \rangle$ with an infinite set $(Y, <)$ of indiscernibles on $U$. As in the proof of Theorem 6.19, we assume $|U^{\mathfrak{M}}| = \kappa$. Without loss of generality, we have $|Y| \geq \kappa$, so we pick a subset $X \subseteq Y$ of size $\kappa$. We generate a substructure $\mathfrak{N} \preccurlyeq_{\mathcal{F}} \mathfrak{M}$ of size $\kappa$ with $X \subseteq N$. Hence, $\mathfrak{N}$ is of the form $\langle N, U^{\mathfrak{N}}, \dots \rangle$ and has type $(\kappa, \kappa)$, that is $|U^{\mathfrak{N}}| = \kappa$, and $\mathfrak{N} \models T$. From Theorem 6.20, $T$ admits $(\mu, |U|)$ for any $\mu \geq \kappa$, i.e, it admits $(\mu, \kappa)$ for every $\mu \geq \kappa$, as desired. □

To finish the paper we establish some inequalities. Since $\mathfrak{a}(\kappa^+)$ is the first ordinal not in $\mathbf{OA}_{\kappa^+}$, it is clear that $\kappa^+ \leq \mathfrak{a}(\kappa^+)$. We also have $\mathfrak{h}_\kappa = \mathfrak{m}_\kappa < \beth_{(2^\kappa)^+}$, $\mathfrak{h}_\kappa = \beth_{\mathfrak{a}(\kappa^+)}$. So,

$$\kappa^+ \leq \mathfrak{a}(\kappa^+) < (2^\kappa)^+.$$

This is [Di75, p.273], but we have corrected the proof to reach this result. In case $cf(\kappa) > \omega$, we have $\kappa^+ < \mathfrak{a}(\kappa^+)$ as we next corroborate ([Di75, p. 273, after Remark]). The next construction is well known.

(⊛) We consider an uncountable cardinal $\kappa$ of uncountable cofinality. We define functions $f \in \kappa^\kappa$ for every $\alpha < \kappa^+$, with the following property: For every $\alpha < \beta < \kappa^+$ there is a $\mu < \kappa$ such that $f_\alpha(\gamma) < f_\beta(\gamma)$ for any $\gamma > \mu$. We proceed by recursion on $\eta < \kappa^+$. Let $(\xi_\nu : \nu < cf(\eta))$ be a cofinal sequence in $\eta$. If $cf(\eta) < \kappa$ we set

$$f_\eta(\alpha) = \sup\{f_{\xi_\nu}(\alpha) : \nu < cf(\eta)\}.$$

Otherwise, set

$$f_\eta(\alpha) = \sup\{f_{\xi_\nu}(\alpha) : \nu < \alpha\}$$

for every ordinal limit $\alpha$.

We define an order $\lhd$ among elements $f, g \in \kappa^\kappa$,

$$f \lhd g \qquad \Leftrightarrow \qquad \exists\, \alpha < \kappa \forall\, \beta < \kappa(\alpha < \beta \to f(\beta) < g(\beta)).$$

Let $L \subseteq \kappa^\kappa$ be a linearly $\lhd$-ordered set of size $\kappa^+$.

**Lemma 6.22.** *L is well founded.*

*Proof.* See [BaKu71, Lemma 1.7]. □

We verify that $\kappa^+$ is $\kappa^+$-accessible, that is $\kappa^+ < \mathfrak{a}(\kappa^+)$, by showing that $\kappa^+$ is $\kappa^+$-semi-accessible (Theorem 6.11). In line with Definition 6.10, we set $L = D$. Notice that $\kappa \times \kappa \in H_{\kappa^+}$, as well as the functions $f \in L$, $D \subseteq \wp(\kappa \times \kappa)$,

$$D(f) \Rightarrow Fun(f) \wedge dom(f) = \kappa \wedge ran(f) \subseteq \kappa.$$

Furthermore, there exists $F \in \kappa \times \kappa$ with

$$Fun(F) \wedge Bij(f) \wedge \forall\, f \in D \exists\, \alpha \in \kappa(F(\alpha) = f).$$

The members of $L$ follows (⊛). The order, $\lhd$ is clearly $\Sigma_0$ in $\kappa$, and $(D, \lhd)$ is a linearly ordered set. In view of Lemma 6.22, $(D, \lhd)$ is well-ordered, and

$$\kappa^+ \leq h(D, \lhd).$$

Hence, $\kappa^+$ is $\kappa^+$-semi-accessible, so it is $\kappa^+$-accessible.

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