

SOME DEVELOPMENTS ON THE INFINITARY LOGIC $L_{\kappa\omega}$

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ABSTRACT. Let κ be a regular cardinal. We first lower the bound of the Hanf number for $L_{\kappa\omega}$ -theories with the JEP, or with prime models and amalgamation. We also treat the two cardinal problem. Next, we work with fragments \mathcal{F} of $L_{\kappa^+\omega}$ to investigate \mathcal{F} -isomorphism of homogeneous models, categoricity, and the size of sets of \mathcal{F} -types realized in such models. We establish a relationship between prime and atomic models, and prove isomorphism of prime models for some fragments of $L_{\kappa\omega}$. Finally, we use another consistency property to provide a direct proof that the well-order is not definable in $L_{\kappa\omega}$ even as an RPC class.

1. INTRODUCTION

The infinitary logic $L_{\kappa\omega}(\mathcal{L})$ contains the first-order logic $L_{\omega\omega}(\mathcal{L})$ and allows conjunctions and disjunctions of set of formulas of size less than κ , where \mathcal{L} is a first order language of cardinality at most κ . The codes of $L_{\kappa\omega}(\mathcal{L})$ -formulas are sets in H_{κ^+} . Suppose that \mathbb{A} is a transitive set. Then, $\mathcal{F} = \mathbb{A} \cap H_{\kappa^+}$ is a fragment of $L_{\kappa\omega}(\mathcal{L})$ when \mathbb{A} satisfies convenient closure properties. In particular, this happens when \mathbb{A} is an admissible set or a primitive recursive (pr) closed set or a model of stronger theories like ZFC^- . Any admissible set is pr closed, but, for instance, if α is an admissible ordinal, the first pr closed ordinal above α is not admissible. In fact, there are plenty of non-admissible pr closed ordinals. If z is a set, it is easy to describe how to construct the least pr closed set containing z (the pr closure of z), whereas achieving an easy description of the admissible closure can be challenging.

In the work [Ke71], Keisler extends numerous results from $L_{\omega\omega}$ to $L_{\omega_1\omega}$. He introduces countable fragments of $L_{\omega_1\omega}$ with very weak properties, and also admissible fragments. Frequently, he works with the latter ones or strengthens his original fragments in order to ensure some characteristics. When we want to extend results from $L_{\omega_1\omega}$ to $L_{\kappa\omega}$, $\kappa > \omega_1$, we realize that this is not a trivial task. To start, we need a suitable consistency property for our logic, and we look at the consistency properties of Karp-Green ([Gr72] and [Gr75]). It is not just a matter of adjusting items, but developing absolutely new techniques and ideas in order to apply these tools. It is clear that there is a wealth of details to deal with. Several results that we thought have already been proved do not fit in our context, because either they are designed only for admissible fragments, or they

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are not completely developed. So, we correct, improve, and establish new results in order to obtain the correct generalizations.

We will work with pr closed fragments of $L_{\kappa\omega}$. This paper strongly relies on, and should be seen as a sequel to, [GaVaVi]. First, we improve the Hanf number computation for theories in a fragment \mathcal{F} of $L_{\kappa\omega}$ of size κ , when the \mathcal{F} -theories have the JEP property. We also deal with other scenarios like the amalgamation property and the two cardinal versions of such computations. After that, we investigate homogeneous models. We succeed in extending the results of Keisler to $L_{\kappa\omega}$ like isomorphy of homogeneous models satisfying the same types. We also examine categoricity and the size of the set of types realized in some models. Next, we obtain results about atomic and prime models in $L_{\kappa\omega}$, like those of Keisler in $L_{\omega_1\omega}$. Our main result establishes a relationship between atomic and prime models, whose proof requires a subtle use of a consistency property due to Green-Karp. Finally, we use another consistency property of Green to show that the well-orders are not an RPC class. Here, we obtain a double gain: we find a more direct proof of this undefinability property, and we achieve the result under a weaker hypothesis than in [Di75].

2. PRELIMINARIES

We use Gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}, \dots$ to denote structures and the corresponding roman letters A, B, M, \dots to represent their universes. We assume acquaintance with notions of infinitary logic, as well as their model theory. We refer the reader to [Ke71] or [Di75] for such concepts. If C is a set of new constants, $\mathcal{L}(C)$ denotes the language obtained by adding the constants in C to \mathcal{L} . Then, we assemble the infinitary logic $L_{\kappa\alpha}(C)$. Given a first order language \mathcal{L} , the set of formulas of $L_{\kappa\alpha}(\mathcal{L})$ will be denoted by $Fml(L_{\kappa\alpha}(\mathcal{L}))$. We will focus on infinitary logics $L_{\kappa\omega}$, where the cardinal κ is bigger than ω_1 and regular. If \mathfrak{A} is an \mathcal{L} -structure and $D \subseteq A$, (\mathfrak{A}, D) means that we are working with the extended language $\mathcal{L}(D)$, and for each $d \in D$, the constant symbol \dot{d} is interpreted in \mathfrak{A} as the element d .

If x is a set, $TC(x)$ is its transitive closure. For a cardinal λ , H_λ is the collection of sets of hereditary cardinality less than λ . Whenever we deal with $L_{\kappa\omega}(\mathcal{L})$, we assume that \mathcal{L} is a first-order language of regular cardinality less than κ . For any unexplained notions see [GaVaVi].

3. FRAGMENTS

A function $f : V^n \rightarrow V$ is called *primitive recursive* (pr) if it is generated by successive applications of the following schemata:

- i) $f(x_1, x_2, \dots, x_n) = x_i, 1 \leq i \leq n$.
- ii) $f(x_1, x_2, \dots, x_n) = \{x_i, x_j\}$, where $1 \leq i, j \leq n$.
- iii) $f(x_1, x_2, \dots, x_n) = x_i - x_j$, where $1 \leq i, j \leq n$.
- iv) $f(x_1, x_2, \dots, x_n) = h(g_1(x_1, x_2, \dots, x_n), \dots, g_k(x_1, x_2, \dots, x_n))$, where g_1, \dots, g_k, h are pr functions.
- v) $f(y, x_1, x_2, \dots, x_n) = \bigcup_{z \in y} g(z, x_1, x_2, \dots, x_n)$, where g is a pr function.
- vi) $f(y, x_1, x_2, \dots, x_n) = g(y, x_1, x_2, \dots, x_n, \langle f(z, x_1, x_2, \dots, x_n) \mid z \in y \rangle)$, where g is a pr function.

For more about pr functions, see [JeKa71].

A relation $R \subset V^n$ is pr, if its characteristic function is a pr function. The function $TC(x)$ (transitive closure of x), and the relations $dom(x)$ (domain of x) and $ran(x)$, among others, are pr. The set x is pr closed if for any pr function $f : V^n \rightarrow V$ and every $a_1, \dots, a_n \in x$, we have $f(a_1, \dots, a_n) \in x$. With $prC(z)$ we denote the pr closure of a set z . If z is a set, then

$$prC(z) = \{f(u) : u \in z, f \text{ is a pr function}\}.$$

Hence, for infinite sets z , $|prC(z)| = |z|$.

The set $Fml(L_{\kappa\omega}(\mathcal{L}))$ of $L_{\kappa\omega}(\mathcal{L})$ -formulas is the smallest set which contains all primitive formulas, and it is closed under $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and under the conjunction and disjunction of fewer than κ formulas and quantification over fewer than ω variables (in fact, $L_{\omega\omega}(\mathcal{L}) \subseteq L_{\kappa\omega}(\mathcal{L})$). In what follows, whenever we state \mathcal{F} is a fragment, we mean \mathcal{F} is a pr closed fragment. The reader is addressed to [Ka68] for details on codifications of our formulas. We always assume that $Fml(L_{\kappa\omega}(\mathcal{L}))$ consists of codes of formulas.

Formally, we define fragment.

Definition 3.1. \mathcal{F} is called a *fragment* of $L_{\kappa\omega}(\mathcal{L})$ (or a $L_{\kappa\omega}(\mathcal{L})$ -fragment), if it contains the set of $L_{\omega\omega}(\mathcal{L})$ and there exists a non-empty transitive pr closed set A such that $\mathcal{F} = Fml(L_{\kappa\omega}) \cap A$.

With $Th_{\mathcal{F}}(\mathfrak{M})$ we denote the complete theory of the \mathcal{L} -structure \mathfrak{M} in the fragment $\mathcal{F}(M)$.

Remark 3.2. We notice that our fragments have the following stronger closure properties, because they are based upon pr closed transitive sets.

- If x is a variable and $\varphi \in \mathcal{F}$, then $\forall x\varphi, \exists x\varphi \in \mathcal{F}$.
- If $f = \langle \varphi_i \mid i < \gamma \rangle \subseteq \mathcal{F}$, then $\bigvee f, \bigwedge f \in \mathcal{F}$.
- If $\exists \vec{v} \bigvee_{\psi \in X} \psi(\vec{v}) \in \mathcal{F}$, then $\bigvee_{\psi \in X} \exists \vec{v} \psi(\vec{v}) \in \mathcal{F}$.
- If $(\bigvee \Phi) \wedge \theta \in \mathcal{F}$ then $\bigvee \{ \varphi \wedge \theta \mid \varphi \in \Phi \} \in \mathcal{F}$.

Our fragments share suitable characteristics.

Proposition 3.3. *Let κ be a regular cardinal. Let \mathcal{F} be a fragment of $L_{\kappa\omega}$ of regular size less than κ . Then, $\mathcal{F} \in H_{\kappa}$. In particular, if \mathcal{F} is a fragment of $L_{\kappa\omega}$, such that $|\mathcal{L}| \leq |\mathcal{F}| = \lambda < \kappa$, then $\mathcal{F} \in H_{\kappa}$.*

Proof. We proceed by induction on the complexity of the formulas to prove that if $\varphi \in \mathcal{F}$, then $\varphi \in H_{\kappa}$, so $\mathcal{F} \subseteq H_{\kappa}$. Hence, $\mathcal{F} \in H_{\kappa}$ since $|\mathcal{F}| < \kappa$. \square

Definition 3.4. Given a formula φ of $L_{\kappa\omega}(C)$, $\sim \varphi$ denotes the formal negation of φ . This negation is defined by induction on the complexity of φ as follows.

$$\begin{aligned} \sim \varphi &\equiv \neg \varphi \text{ if } \varphi \text{ is atomic ;} \\ \sim (\neg \varphi) &\equiv \varphi; \\ \sim \left(\bigwedge \Phi \right) &\equiv \bigvee \{ \neg \varphi : \varphi \in \Phi \}; \\ \sim \left(\bigvee \Phi \right) &\equiv \bigwedge \{ \neg \varphi : \varphi \in \Phi \}; \\ \sim (\forall x \varphi) &\equiv \exists x \neg \varphi; \\ \sim (\exists x \varphi) &\equiv \forall x \neg \varphi. \end{aligned}$$

It is clear that $\sim \varphi$ is logically equivalent to $\neg \varphi$.

Proposition 3.5. *For every set $\Theta \subset Fml(L_{\kappa\omega}(\mathcal{L}))$ of regular size less than κ , there exists a fragment \mathcal{F} of $L_{\kappa\omega}(\mathcal{L})$ such that $\Theta \subset \mathcal{F}$ and $|\mathcal{F}| = \max\{|\Theta|, |\mathcal{L}|\} < \kappa$.*

Proof. We consider Θ as a subset of H_{κ} . Then, Θ belongs to H_{κ^+} . We build $A = \bigcup_{i < \omega} A_i$ in H_{κ^+} as follows:

$$\begin{aligned} A_0 &= \Theta \cup PFml(L_{\kappa\omega}(\mathcal{L})) \\ B_{n+1} &= TC(A_n) \\ A_{n+1} &= prC(B_{n+1}) \end{aligned}$$

for all $n < \omega$. Clearly, $A = \bigcup_{n \in \omega} A_n$ is transitive and pr closed, hence $\mathcal{F} = L_{\kappa\omega}(\mathcal{L}) \cap A$ is the desired fragment.

It is worth noticing that $A \in H_{\kappa^+}$. \square

Remark 3.6. Let κ be a regular cardinal. Let \mathcal{F} be a fragment of $L_{\kappa\omega}(\mathcal{L})$ whose cardinality is $\lambda < \kappa$. Let C be a set of new constants of cardinality λ . As above, we can generate a fragment $\mathcal{F}(C) \supseteq \mathcal{F}$ (we also denote it with \mathcal{F}_C) such that, if $\varphi(\vec{x}) \in \mathcal{F}$ and $\vec{c} \in C$, then $\varphi(\vec{c}) \in \mathcal{F}(C)$ and $|\mathcal{F}(C)| = \lambda$.

Lemma 3.7. *Let $\mathcal{F} = Fml(L_{\kappa\omega}(\mathcal{L})) \cap A$ be a fragment, where A is a pr closed set. The following assertions hold.*

- (1) $\omega \subset A$.
- (2) \mathcal{F} is closed under negation, finite disjunction, and conjunction.
- (3) If $\varphi \in \mathcal{F}$ and $\vec{x} \in A$ are variables, then $\forall \vec{x} \varphi \in \mathcal{F}$ and $\exists \vec{x} \varphi \in \mathcal{F}$.
- (4) If $\varphi \in \mathcal{F}$, every subformula of φ belongs to \mathcal{F} .
- (5) If $\theta(v_1, \dots, v_n)$ is an atomic \mathcal{L} -formula and $x_1, \dots, x_n \in A$, then $\theta(x_1, \dots, x_n) \in \mathcal{F}$.
- (6) If Θ is a set of formulas of size less than κ with $\Theta \in A$, then $\bigwedge \Theta \in \mathcal{F}$ and $\bigvee \Theta \in \mathcal{F}$.
- (7) If $\varphi \in \mathcal{F}$, then $\sim \varphi \in \mathcal{F}$.
- (8) If $\varphi \in \mathcal{F}$, there exists a variable $x \in A$ which does not occur in φ .

Proof. All properties follow from the definition of \mathcal{F} . \square

Now, we introduce the notion of abstract logic. Although we shall treat only infinitary logics, we prefer to provide a broader context for some definitions. See [BaFe85] for an extensive treatment of abstract logic.

Definition 3.8. A logic is a pair $(\mathbb{L}, \models_{\mathbb{L}})$, where \mathbb{L} is an application with domain languages \mathcal{L} in such a way that $\mathbb{L}[\mathcal{L}]$ is the class of \mathbb{L} -sentences of the language \mathcal{L} , and the \mathbb{L} -satisfaction relation is a relation between structures and \mathbb{L} -sentences.

With $\mathbb{L}_{\omega\omega}(\mathcal{L})$ or simply $L_{\omega\omega}(\mathcal{L})$ we denote the first order logic, $\mathfrak{A} \in Str[\mathcal{L}]$ means that \mathfrak{A} is an \mathcal{L} -structure. If $\varphi \in \mathbb{L}[\mathcal{L}]$, we write $Mod_{\mathbb{L}}^{\mathcal{L}}(\varphi)$ or simply $Mod(\varphi)$, \mathcal{L} and \mathbb{L} are given, for the collection $\{\mathfrak{A} \in Str[\mathcal{L}] : \mathfrak{A} \models_{\mathbb{L}} \varphi\}$; and for $\Phi \cup \{\varphi\} \subseteq \mathbb{L}[\mathcal{L}]$, the \mathbb{L} -consequence relation is defined by

$$\Phi \models_{\mathbb{L}} \varphi \Leftrightarrow \text{for all } \mathfrak{A} \in Str[\mathcal{L}], \quad \mathfrak{A} \models_{\mathbb{L}} \Phi \Rightarrow \mathfrak{A} \models_{\mathbb{L}} \varphi.$$

Two structures $\mathfrak{A}, \mathfrak{B}$ are \mathbb{L} -equivalent, $\mathfrak{A} \equiv_{\mathbb{L}} \mathfrak{B}$, if and only if for all $\varphi \in \mathbb{L}[\mathcal{L}]$,

$$\mathfrak{A} \models_{\mathbb{L}} \varphi \Leftrightarrow \mathfrak{B} \models_{\mathbb{L}} \varphi.$$

We write $Th_{\mathbb{L}}(\mathfrak{A})$ for $\{\varphi \in \mathbb{L}[\mathcal{L}_{\mathfrak{A}}] : \mathfrak{A} \models_{\mathbb{L}} \varphi\}$, the \mathbb{L} -theory of \mathfrak{A} , where $\mathcal{L}_{\mathfrak{A}}$ is the language of \mathfrak{A} .

Let \mathfrak{C} be an \mathcal{L} -structure; the subset $D \subseteq C$ is \mathcal{L} -closed, when for every constant symbol $c \in \mathcal{L}$, $c^{\mathfrak{C}} \in D$, and for every symbol of n -function $f \in \mathcal{L}$, and any n -tuple $\vec{a} \in D$, we have $f(\vec{a}) \in D$. There could not be a relationship between $L_{\omega\omega}$ and an abstract logic $\mathbb{L}[\mathcal{L}]$, but we work with regular logics, which satisfy many closure properties that ensure $L_{\omega\omega}[\mathcal{L}] \subseteq \mathbb{L}[\mathcal{L}]$. Amongst such closure properties, we have the relativization property that will be important in the last endeavor of this paper.

A logic has the relativization property when the following holds. If $U \notin \mathcal{L} \cup \mathcal{L}'$, $\chi \in \mathbb{L}[\mathcal{L}' \cup \{U\}]$ and $\varphi \in \mathbb{L}[\mathcal{L}]$, then there is a sentence $\psi \in \mathbb{L}[\mathcal{L} \cup \mathcal{L}']$ such that for all $(\mathcal{L} \cup \mathcal{L}')$ -structures \mathfrak{B} , if the set $\chi^{\mathfrak{B}} = \{b \in B : (\mathfrak{B}, b) \models \chi\}$ is \mathcal{L} -closed in \mathfrak{B} , then

$$\mathfrak{B} \models \psi \Leftrightarrow (\mathfrak{B} \upharpoonright \mathcal{L}) \upharpoonright \chi^{\mathfrak{B}} \models \varphi.$$

So ψ is the relativization of φ to $\{x : \chi(x)\}$, often written φ^U , if $\chi = U(x)$.

- (c) Let $\{t_i : i \in I\}$ be a set of closed terms, with power $< \kappa$ and $< \omega$ constants from D . Fix $I_1 = \{i \in I : t_i \notin D\}$ and $I_2 = I \setminus I_1$. Fix $\{e_i : i \in I\}$ a set of constants none of which appear in σ_0 or $\{t_i : i \in I\}$. Thus for $(\mathfrak{A}, b_1, \dots, b_n) \models \bigwedge \sigma_0[s]$, $\sigma'_0 = \sigma_0 \cup \{e_i = t_i : i \in I_1\}$, $s' = (s \setminus \{(e_i, s(e_i)) : i \in I_1\}) \cup \{(e_i, s(t_i)) : i \in I_1\}$ we have

$$(\mathfrak{A}, b_1, \dots, b_n) \models \bigwedge \sigma'_0[s'].$$

As $t_i \in D$ for any $i \in I_2$ and I_2 is finite, we are able to find interpretations for e_i . If t_i appears in σ_0 , say $d_{r_{j(i)}}$, then for $s'' = (s' \setminus \{(e_i, s(e_i)) : i \in I_2\}) \cup \{(e_i, b_{j(i)}) : i \in I_2\}$. If t_i does not appear in σ_0 then t_i is some d_r . As in C7(b), we can locate r among the r_j 's and interpret e_i accordingly.

Take ψ a sentence stating that $<$ has a transitive linear order of $\chi(x)$. We claim that

$$\underbrace{\{\psi, \varphi\} \cup \{d_{r_0} < d_{r_1} : r_0 < r_1\} \cup \{\chi(d_r) : r \in \mathbb{Q}\}}_{\sigma_1} \in \Sigma.$$

Take $\alpha < 2^{*\kappa}$ then there is some $\mathfrak{A} \models \varphi \wedge \psi$ with $(\alpha, <) \subset (\chi(\mathfrak{A}), <)$. For any $\xi < \alpha$, we take a new constant e_ξ . If

$$(\mathfrak{A}, \xi)_{\xi < \alpha} \models \underbrace{\bigwedge \{e_\xi < e_\zeta : \xi < \zeta < \alpha\} \wedge \bigwedge \{\chi(e_\xi) : \xi < \alpha\} \wedge \varphi \wedge \psi}_{\varphi_0},$$

using downward Löwenheim-Skolem theorem we can find a model \mathfrak{A}_α for φ_0 with power $< 2^{*\kappa}$. Thus the set of ordinals in $\chi(\mathfrak{A}_\alpha)$ is less than $2^{*\kappa}$, we conclude $\mathfrak{A}_\alpha \models \psi \wedge \varphi$.

Now, there is a model \mathfrak{B} for φ , and this defines a linear order $(\chi(\mathfrak{B}), <)$ that has a copy of the rationals. \square

Theorem 7.5. *The well-order is not $\text{RPC}(\mathcal{L}_{\infty\omega})$ -definable.*

Proof. Suppose that $\varphi, \chi(x) \in \mathcal{L}_{\kappa\omega}(\mathcal{L}^+)$ make an RPC-definition to the well-ordered structures. That is $(A, <)$ is well-order if and only if for some \mathcal{L}^+ -expansion \mathfrak{A}^+ of $(A, <)$, and $\mathfrak{A}^+ \models \varphi$ holds exactly when $(\chi(\mathfrak{A}^+), <) = (A, <)$ is well-ordered. For any $\alpha < 2^{*\kappa}$ take a \mathcal{L}^+ -expansion \mathfrak{A}_α^+ of $(\alpha, <)$. Thus \mathfrak{A}_α^+ satisfies the hypothesis of Theorem 7.4. Therefore, there is $\mathfrak{B} \equiv_{\mathcal{L}_{\kappa\omega}} \mathfrak{A}_\alpha^+$ such that $\chi(\mathfrak{B})$ is not well-ordered. But $\mathfrak{B} \models \varphi$, which is a contradiction. \square

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