A REPRESENTATION OF RECURSIVELY ENUMERABLE SETS THROUGH HORN FORMULAS IN HIGHER RECURSION THEORY

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ABSTRACT. We extend a classical result in ordinary recursion theory to higher recursion theory, namely that every recursively enumerable set can be represented in any model $\mathfrak{A}$ by some Horn theory, where $\mathfrak{A}$ can be any model of a higher recursion theory, like primitive set recursion, α-recursion, or β-recursion. We also prove that, under suitable conditions, a set defined through a Horn theory in a set $\mathfrak{A}$ is recursively enumerable in models of the above mentioned recursion theories.

1. INTRODUCTION

The study of recursion theory on the ordinals was started by Takeuti, Kripke, Platek, Kreisel and Sacks. Barwise developed an extensive theory of admissible structures, where α-recursion theory takes place. The analogy with classical recursion theory has been quite striking. Many theorems, especially some about recursively enumerable sets have been successfully extended to all admissible ordinals.

One general program of higher recursion theory is to enlarge or "lift" results from classical recursion to ordinal recursion theory, in particular to α- or admissible recursion theory. But there are other kinds of higher recursion which also draw our attention.

In [JenKar] the authors develop the theory of primitive recursive set functions and they give a perfectly good theory of this set of functions on any transitive primitive recursively closed class. Some of the simplest results of ordinary recursion theory can be generalized to the context of an arbitrary

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primitive recursively closed ordinal. In order to generalise the deeper theorems it has often been found convenient to impose additional hypotheses on the ordinal involved. One of the most popular assumptions is *admissibility*.

Every primitive recursive set function has a Σ_1 definition in many important transitive classes, like Gödel's constructible universe L. Actually, every admissible set is primitive recursively closed. In fact, admissibility is a stronger condition than primitive recursively closedness, since an admissible set M can contain numerous primitive recursively closed elements.

In recent years there has been a renewed interest on higher recursion theory due, in part, to the need to extend the notion of computability (see for example [HaLe00], [KoSe09], [Ko07]). Along with this line of thought, higher recursion theory is still a major place to generalise results from classical recursion, which is what we are concerned with.

This paper is devoted to the following matter. In [Smu61], Smullyan introduced an elegant development of ordinary recursion theory using a kind of axiomatic system of derivation, the so-called *elementary formal systems*. Among other things he proved that "every recursively enumerable (r.e.) subset of the natural numbers can be represented by some elementary formal system". Later on, it has been realized that some results of Logic Programming were already obtained in [Smu61] simply because elementary formal systems are precisely what now is known as *Horn clause programs*. Generalisations of Smullyan's result for n-ary relations on $\mathbb{N}$ can be found in the literature, see for instance [Doe94] or [Hod93], where the proofs are given using logical programs or models of computing machines, respectively.

As we already mentioned one of the first tasks of the higher recursion theory was to extend deep results in ordinary recursion theory. Our goal is precisely to extend Smullyan's result [Smu61] to higher recursion theory. For a modern statement of this theorem see [Hod93, Theorem 9.0.1, p.480] and [Doe94]. Moreover our result partially answers questions (2) and (3) of [Fit81, p. 294].

Since the proofs given in [Hod93] and [Doe94] of the classical result rely on logical programs or models of computing machines, we cannot hope to use modified versions of them to verify the result in higher recursion. Thus we are prompted to find a completely new proof.

Actually we are able to extend both directions of Smullyan's result to arbitrary primitive recursively closed sets, not only to those which are admissible, and also for some structures involved in β-recursion theory.

Now this is a good point to formulate the classical result of Smullyan:

Theorem 1.1. *Let X be an r.e. set of natural numbers, then there are a finite language $\mathcal{L}$ (including a unary predicate symbol R, a unary function symbol s and a constant symbol $\emptyset$) and a finite Horn clause theory* $\mathbf{T}$ *such* *that for every natural number n,*

$$(\maltese) \qquad n \in X \quad \Leftrightarrow \quad \mathbf{T} \models R(s^n(\emptyset)).$$

Conversely, if X is a set of natural numbers such that ($\maltese$) holds for some first order finite theory **T***, then X is a r.e. set.*

So, the first challenge to extend Smullyan's result to higher recursion is to incorporate ordinals bigger than ω. To this end, we have to devise a theory **T** (Horn too) which allows us to handle bigger ordinals. This is done in section 3. In section 4 we introduce admissible structures, primitive recursively closed structures and those used in β-recursion theory. In section 5 we derive the converse direction of our main theorem for higher recursion theories. Finally, in section 6 we establish our main results for classical recursion theory.

2. PRELIMINARIES

In this section we give the basic definitions that we use throughout.

We will work with a finite first order language $\mathcal{L}$ which includes $\in$. We use the following notation

$$f[x] = \{f(y) : y \in x\}.$$

On is the class of ordinals. Our background theory is ZFC. With V we denote the universe of sets.

We begin with the definition of Horn formula.

Definition 2.1. Let $\mathcal{L}$ be a first order language. The $\mathcal{L}$-formula φ is a *basic Horn formula* if it is of the form

$$\theta_1 \wedge \cdots \wedge \theta_m \to \psi,$$

where ψ is either an atomic formula or *falsum* $\perp$ and the θ_i are atomic formulas. A *Horn formula* is a finite (possibly empty) string of quantifiers, followed by a conjunction of basic Horn formulas. Horn clauses are of the form

$$\forall x_1, \cdots, x_l \sigma,$$

where σ is a basic Horn formula. A Horn clause theory is a set of Horn clauses.

We now define the Σ_1-formulas for the language of set theory LST, which includes only $\{\in, =\}$.

Definition 2.2. Let LST be our language. The Δ_0 ($= \Sigma_0 = \Pi_0$)-formulas of LST are defined recursively as follows:

(1) Every atomic formula is a Δ_0-formula.
(2) If φ and ψ are Δ_0-formulas, so are $\varphi \wedge \psi$ and $\neg\varphi$.

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