THE NATURE OF JÓNSSON CARDINALS

LUIS MIGUEL VILLEGAS SILVA

ABSTRACT. This paper is about Jónsson cardinals in three different aspects. First, we extend a result of Silver on Jónsson and measurable cardinals. Second, we generalize a theorem of Mitchell to show that if $A \subseteq \kappa$, then in some cases J_κ^A is a Jónsson algebra for a singular cardinal κ. Third, we provide a lower bound for the Hanf number of the infinitary logic $L_{\lambda\omega_1}$ using Jónsson cardinals, for some cardinals λ.

1. INTRODUCTION

Jónsson cardinals are named after Bjarni Jónsson, who in 1962 asked whether or not every algebra of cardinality κ has a proper subalgebra of the same size. The cardinals κ with such a subalgebra are now called Jónsson cardinals.

Jónsson cardinals are strange entities. They may be conceived as large cardinals or not. First of all, ω is not Jónsson. It is an open question whether $\aleph_\omega$ can be Jónsson; if it is, it would be the least one. So, Jónsson cardinals can be singular, and under GCH all of them are limits. Secondly, Jónsson cardinals are incompatible with $V = L$; they imply the existence of $a^\#$ for any $a \subseteq \omega$. Ramsey cardinals are Jónsson. Thirdly, the definition of Jónsson and Rowbottom cardinals can be given so as to follow the template of asserting some arrow relations.

An outstanding result of J. Silver is that if $\aleph_\omega$ is Jónsson and $2^{\aleph_0} < \aleph_\omega$, then $\aleph_\omega$ becomes measurable in an inner model. It is natural to ask if there is a corresponding claim for bigger singular Jónsson cardinals, in case they exist. We shall extend the theorem of Silver to the following. If κ is a singular Jónsson cardinal and $2^{cf(\kappa)} < \kappa$, then κ is measurable in an inner model.

The second part of the paper sets out to generalize a theorem of Mitchell. We shall show that if κ is a singular cardinal and $A \subseteq \kappa$, then J_κ^A is not a Jónsson algebra, under some natural conditions on $J_{\kappa^+}^A$. As a consequence, for such a κ there are no Jónsson Cardinals $\lambda \geq \kappa$ in J^A.

The phenomenon of the effect of large cardinals on the Hanf number of fragments of infinitary logics deserves some attention. Fifty years ago, Kunen gave a lower bound for the Hanf number $h(L_{\omega_1\omega_1})$ of $L_{\omega_1\omega_1}$. Namely, when there exists a κ-complete normal ultrafilter U on κ and assuming $V = L[U]$, he proved that $h(L_{\omega_1\omega_1}) > \kappa$. His proof is based on iterated ultrapowers. We shall prove that if κ is a Jónsson cardinal, then in $V = L[A]$ for many $A \subseteq \kappa^+$, $h(L_{\kappa^{++}\omega_1}) > \kappa^{(n)}$ for every $n \in \mathbb{N}$.

Next section collates notation and results about Rowbottom and Jónsson cardinals. All the mentioned results are known. It can be verified that if a cardinal κ is ν-Rowbottom for some $\nu < \kappa$, then κ is Jónsson. So, we shall take care solely of Jónsson cardinals. It is not the purpose of this paper to survey results on Jónsson cardinals in any detail but to contribute further to the mentioned issues.

In section 3, we go into a generalisation of a theorem of Silver. We show that, if μ is a singular Jónsson cardinal, then under an appropriate hypothesis μ is measurable in an inner model. In section 4 we completely develop the construction of a liftup for J-structures of the

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Since $X \subseteq \kappa$, it is a bounded subset of κ^+, so $X \in J^A_{\kappa^+}$, as we needed.

For $n > 0$, we prove that κ is Jónsson in $J^A_{\kappa^{(n+1)}}$ by the same proof. ❒

Theorem 6.8. *Assume that κ is a Jónsson cardinal and $2^{\underset{\smile}{\kappa^+}} = \kappa^+$. Then,*

$$h(L_{\kappa^{++}\omega_1}) > \kappa^{(n+1)}, \qquad \textit{for each } n < \omega$$

Proof. Assume the Theorem is false, and let $n < \omega$ be the least possible counterexample. Thus,

$$h(L_{\kappa^{++}\omega_1}) \leq \kappa^{(n+1)}$$

By Theorem 6.7 we have a model $\mathscr{U}$ of $\exists v \Psi_n(v)$ of size $\kappa^{(n+1)}$. So, we can pick a model $\mathfrak{M}$ of $\exists v \Psi_n$ of size at least $\kappa^{(n+6)}$ (or bigger, of course). By definition of Ψ_n, $M = J^A$, contains κ and its successors at least until $\kappa^{(n+5)}$. Therefore, the biggest cardinal μ in $\mathfrak{M}$ is bigger than κ^+. Since $A \subseteq \mu$, we can prove, as in the proof of Theorem 5.1, that μ is not Jónsson in $\mathfrak{M}$, a contradiction. ❒

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA IZTAPALAPA, CDMX, MÉXICO, **e-mail: villegas63@gmail.com**