DEVELOPMENTS FROM A CONSISTENCY PROPERTY FOR THE INFINITARY LOGIC $L_{\kappa\omega}$

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ABSTRACT. This paper extends several results holding in the infinitary logic $L_{\omega_1\omega}$ to $L_{\kappa\omega}$ for κ a regular cardinal. We prove an omitting type theorem, that allows us to produce end-extensions of certain models. We also show versions of the cardinal transfer problem, which improves the consistency strength of the Gap-1 problem (regular case) in first order logic.

1. INTRODUCTION

In the beginning, classical logic was designed to be the "natural" language to formalize mathematics. Nevertheless, many kinds of mathematical structures cannot be axiomatized through it. Infinitary logic, as an abstract logic, provides a significant extension on expressibility. However, it lacks, in general, of compactness or Upward Löwenheim-Skolem theorem, which are recurrent tools in $L_{\omega\omega}$, first order logic. Loss of compactness has been primarily overcome through consistency properties. A consistency property is an abstraction of The Henkin construction to prove completeness in first order logic. Makkai [Ma69] introduced the notion for admissible logics. In [Gr75] Karp and Green formulated consistency properties in $L_{\kappa\omega}(\mathscr{L})$ for certain cardinals $\kappa > \omega$.

If solving the Two-Cardinal Transfer Theorem for infinitary logic is to be achieved, then Green-Karp consistency properties is a convenient tool to consider. After the works of Karp, Green and Cunningham on consistency properties in the early 70's, there has not been any improvement in this direction. Also, apart from the work of Keisler on $L_{\omega_1\omega}$, we could not record any results around the two-cardinal transfer theorem in $L_{\kappa\omega}$. In order to progress, we extend his results to $L_{\kappa\omega}$ applying the Green's consistency property. While the thrust of Green's notion was mostly completeness for her particular proof system, we privilege the semantic part. To this end, we first show an omitting type theorem. This feature opens the way to building end-extensions. With this theorem at hand, we can manage to solve several instances of the Gap-n cardinal transfer theorem. In this paper we handle the Gap-1 problem in the regular case, which does not require sophisticated combinatorial principles like morasses, as for the Gap-2 (or bigger n) problem. But we do require a model theoretic restriction, i.e. we ask for a theory satisfying the amalgamation property. It turns out, that this is not a decisive restriction. As we shall see, a convenient representation in the original model of the two cardinal transfer theorem allows us to overcome this apparent limitation.

As it has been mentioned before, the Upward Löwenheim-Skolem Theorem fails, in general, in infinitary logics. Therefore, in this paper the treatment of the cardinal transfer problems for

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those logics will be primarily downwards. However, the proof does work also upwards not only for first order but for infinitary theories with arbitrarily large models as well.

We will present the proof of

$$(\circledast) \qquad \langle \kappa, \mu \rangle \to_{\mathscr{F}} \langle \lambda^+, \lambda \rangle,$$

where $\kappa > \mu \geq \lambda$, λ is a regular cardinal and $\mathscr{F}$ is a a fragment of $L_{\kappa\omega}(\mathscr{L})$ of size $\leq \lambda$ with some additional conditions on $\mathscr{F}$. We define the notion of fragment in Section 2. When $\mathscr{F}$ is the first order fragment, our result improves the Chang's Gap-1 Theorem.

As in Keisler's work on

$$\langle \kappa, \mu \rangle \to \langle \aleph_1, \aleph_0 \rangle$$

for countable fragments of $L_{\omega_1\omega}$, we need end-extensions, this time in $L_{\lambda\omega}(\mathscr{L})$ for the transfer problem ($\circledast$).

2. Preliminary concepts and notation

We use Gothic letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{M}$, ... to denote structures and the corresponding roman letters $A, B, M, \ldots$ to represent their universes.

We assume acquaintance with notions of infinitary logic, as well as their model theory. We refer the reader to [Ke71] or [Di75] for such concepts. Given cardinals κ and α, the infinitary logic $L_{\kappa\alpha}(\mathscr{L})$ allows the conjunction and disjunction of fewer than κ formulas and quantification over fewer than α variables. If C is a set of new constants, $\mathscr{L}(C)$ denotes the language obtained by adding the constants in C to $\mathscr{L}$. Then, we can consider the infinitary logic $L_{\kappa\alpha}(\mathscr{L}(C))$. The set of formulas of $L_{\kappa\alpha}(\mathscr{L})$ will be denoted by $Fml(L_{\kappa\alpha}(\mathscr{L}))$. We will focus on infinitary languages $L_{\kappa\omega}(\mathscr{L})$, where the cardinal κ is uncountable and regular. If $\mathfrak{A}$ is an $\mathscr{L}$-structure and $D \subseteq A$, $(\mathfrak{A}, D)$ means that we are working with the extended language $\mathscr{L}(D)$, and for each $d \in D$, the constant symbol $\dot{d}$ is interpreted in $\mathfrak{A}$ as the element d. With $Th_{\mathscr{F}}(\mathfrak{M}, M)$ we denote the complete theory of the $\mathscr{L}$-structure $\mathfrak{M}$ in the fragment $\mathscr{F}$.

If x is a set, $TC(x)$ is its transitive closure. For a cardinal λ, H_λ is the collection of sets of hereditary cardinality less than λ. Whenever we deal with $L_{\kappa\omega}(\mathscr{L})$, unless otherwise stated, we assume that $\mathscr{L}$ is a first order language of regular cardinality less than κ.

2.1. **Fragments.**

There are several definitions of fragment, which are certain sets of formulas fulfilling some desired closure properties.

Keisler [Ke71] supplies a formal definition for a distinguished set of formulas sharing various "natural" closure properties. His closure requirement leads to a weak set theory. Nevertheless, in most applications Keisler requests stronger closure conditions, like admissible fragments. In this note, we provide a definition of fragment which turns out to be useful. One influence for this choice has been [Gr72].

We have chosen primitive recursive closed sets to define our fragments. Being a primitive recursive closed set is weaker than being admissible. Despite this, our fragments contain the weaker Keisler's. The primitive recursively closed sets have enough closure properties. Moreover, closing a set under primitive recursive functions is easier to describe, and does not increase its cardinality. In addition, they supply enough set of formulas, and we can safely interchange quantifiers and disjunctions.

A function $f : V^n \to V$ is called *primitive recursive* (pr) if it is generated by successive applications of the following schemata:

i) $f(x_1, x_2, \ldots, x_n) = x_i$, $1 \leq i \leq n$.
ii) $f(x_1, x_2, \ldots, x_n) = \{x_i, x_j\}$, where $1 \leq i, j \leq n$.
iii) $f(x_1, x_2, \ldots, x_n) = x_i - x_j$, where $1 \leq i, j \leq n$.
iv) $f(x_1, x_2, \ldots, x_n) = h(g_1(x_1, x_2, \ldots, x_n), \ldots, g_k(x_1, x_2, \ldots, x_n))$, where $g_1, \ldots, g_k, h$ are pr functions.

v) $f(y,x_1,x_2,\ldots,x_n)=\bigcup_{z\in y} g(z,x_1,x_2,\ldots,x_n)$, where g is a pr function.
vi) $f(y,x_1,x_2,\ldots,x_n)=g(y,x_1,x_2,\ldots,x_n,\langle f(z,x_1,x_2,\ldots,x_n)\mid z\in y\rangle)$, where g is a pr function.

This kind of set functions is studied by Jensen and Karp in [JenKar71].

We call $R\subset V^n$ a pr relation, if its characteristic function is a pr function. For our purposes, it is relevant to note that the function $TC(x)$ (transitive closure of x), and the relations $dom(x)$ (domain of x) and $ran(x)$, among others, are pr. The set x is pr closed if for any pr function $f:V^n\to V$ and every $a_1,\ldots,a_n\in x$, it occurs that $f(a_1,\ldots,a_n)\in x$. We let $prC(z)$ denote the pr closure of a set z.

Remark 1. If A is a nonempty pr closed transitive set, it is easily seen that the following is true.

- If $a,b\in A$, then $a\cap b$, $a\times b$, $\{a,b\}$, (a,b), $\bigcup a$ belong to A.
- If $a\in A$ and α is the least ordinal which does not belong to the transitive closure of a, then $\alpha\in A$. Namely, if $\alpha=\bigcup TC(a)\cap Or$, then $\alpha\in A$.

In order to define our fragment we should supply an adequate codification for $L_{\kappa\omega}(\mathscr{L})$-formulas. This is standard; the reader is addressed to [Ka68] for details. Going forward, we always assume that $Fml(L_{\kappa\omega}(\mathscr{L}))$ consists of codes of formulas.

From now on, we will make no distinction between formulas and their codes. The set $Fml(L_{\kappa\omega}(\mathscr{L}))$ of $L_{\kappa\omega}(\mathscr{L})$-formulas is the smallest set which contains all primitive formulas $PFml(L_{\kappa\omega}(\mathscr{L}))$, and it is closed under $\neg,\wedge,\vee,\to,\leftrightarrow$ (in fact, $L_{\omega\omega}(\mathscr{L})\subseteq L_{\kappa\omega}(\mathscr{L})$). Furthermore, it satisfies the following properties.

- If x is a variable and $\varphi\in Fml(L_{\kappa\omega}(\mathscr{L}))$, then $\forall x\varphi,\exists x\varphi\in Fml(L_{\kappa\omega}(\mathscr{L}))$.
- If $f=\langle\varphi_i|i<\gamma\rangle$ and $\varphi_i\in Fml(L_{\kappa\omega}(\mathscr{L}))$ for $i<\gamma<\kappa$, then $\bigvee f,\bigwedge f\in Fml(L_{\kappa\omega}(\mathscr{L}))$.

Definition 2.1. $\mathscr{F}$ is called a *fragment* of $L_{\kappa\omega}(\mathscr{L})$, if it contains the set $PFml(L_{\kappa\omega}(\mathscr{L}))$ and there exists a non-empty transitive pr closed set A such that $\mathscr{F}=Fml(L_{\kappa\omega}(\mathscr{L}))\cap A$.

Observe that, for any first order language $\mathscr{L}$ and each fragment $\mathscr{F}$, $L_{\omega\omega}(\mathscr{L})\subseteq\mathscr{F}$ holds. Concerning our fragments, there are some points to be made

Proposition 2.2. *Let κ be a regular cardinal. Let $\mathscr{F}$ be a fragment of $L_{\kappa\omega}(\mathscr{L})$ of regular size less than κ. Then, $\mathscr{F}\in H_\kappa$. In particular, if $\mathscr{F}$ is a fragment of $L_{\kappa\omega}(\mathscr{L})$, such that $|\mathscr{L}|\leq|\mathscr{F}|=\lambda<\kappa$, then $\mathscr{F}\in H_\kappa$.*

Proof. We proceed by induction on the complexity of the formulas to prove that if $\varphi\in\mathscr{F}$, then $\varphi\in H_\kappa$, so $\mathscr{F}\subseteq H_\kappa$. Hence, $\mathscr{F}\in H_\kappa$ since $|\mathscr{F}|<\kappa$. □

Definition 2.3. Given a formula φ of $L_{\kappa\omega}(\mathscr{L}(C))$, $\sim\varphi$ denotes the formal negation of φ. This negation is defined by induction on the complexity of φ as follows.

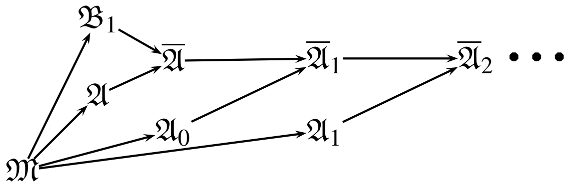
$$\begin{aligned}
\sim\varphi &\equiv \neg\varphi \text{ if } \varphi \text{ is atomic ;}\\
\sim(\neg\varphi) &\equiv \varphi;\\
\sim\left(\bigwedge\Phi\right) &\equiv \bigvee\{\neg\varphi:\varphi\in\Phi\};\\
\sim\left(\bigvee\Phi\right) &\equiv \bigwedge\{\neg\varphi:\varphi\in\Phi\};\\
\sim(\forall x\varphi) &\equiv \exists x\neg\varphi;\\
\sim(\exists x\varphi) &\equiv \forall x\neg\varphi.
\end{aligned}$$

It is clear that $\sim\varphi$ is logically equivalent to $\neg\varphi$.

Lemma 2.4. *Let $\mathscr{F}=Fml(L_{\kappa\omega}(\mathscr{L}))\cap A$ be a fragment, where A is a pr closed set. The following assertions hold.*

(1) $\omega\subset A$.

For the formulas in Λ_2 we proceed as in the case JEP to construct $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ to obtain the following diagram



We set $\mathfrak{B} = \bigcup_{i<\delta} \mathfrak{B}_i$, which is a model of

$$\overline{\sigma} \cup T' \cup \{\psi^{\alpha} : \psi^{\alpha} \in \Psi_{\alpha}, \alpha < \mu_1\} \cup \{\psi^{\gamma} : \psi^{\gamma} \in \Psi_{\gamma}, \gamma < \mu_2\}$$

and the corresponding set belongs to Σ.

Case 3. T has a prime model $\mathfrak{M}$. Let us set $T' = Th_{\mathscr{F}}(\mathfrak{M}, M)$ and replace, as above, T with T' in the definition of Σ. Now it is clear how to proceed.

We have accomplished the task.

□

Corollary 7.5. *Let us suppose that the following holds.*

- κ *is a regular cardinal.*
- $\mathscr{F}$ *is a* κ*-splendid fragment of* $L_{\kappa\omega}(\mathscr{L})$ $|\mathscr{L}| \leq \kappa$.
- T *is an* $\mathscr{F}$*-theory satisfiable with property* $\mho$.
- $\{\Theta_{\alpha}(v_1, \ldots, v_{m_{\alpha}}) \mid \alpha < \kappa, m_{\alpha} < \omega\}$ *is a sequence of sets of* $\mathscr{F}$*-formulas.*
- *For any* $\alpha < \kappa$ *and every* $\mathscr{F}$*-formula* $\psi(v_1, \ldots, v_{m_{\alpha}})$*, if* $T + \exists\vec{v}\psi(\vec{v})$ *has a model, then there is a* $\theta \in \Theta_{\alpha}$ *such that* $T + \exists\vec{v}(\psi(\vec{v}) \wedge \theta(\vec{v}))$ *has a model.*

Assume that either

(i) $\mho$ *is the joint embedding property* ***or***

(ii) $\mho$ *is the amalgamation property,* T *is satisfiable in the* $\mathscr{F}$*-model* $\mathfrak{M}$ *and* $Th_{\mathscr{F}}(\mathfrak{M}, M) \subseteq T$.

(iii) $\mho$ *means:* T *has a prime model.*

Then

$$T + \bigwedge_{\alpha<\kappa} \forall\vec{v} \bigvee_{\theta\in\Theta_{\alpha}} \theta(\vec{v})$$

is satisfiable.

□

8. Further remarks

1. Since we have required always regularity for λ in theorems of the kind

$$\langle\kappa, \mu\rangle \rightarrow_{\mathscr{F}} \langle\lambda^+, \lambda\rangle,$$

it is natural to enquire about results for λ singular. Under the hypothesis $\lambda^{<\lambda} = \lambda + \Box_{\lambda}$, the problem is solved in a forthcoming paper.

2. Concerning the gap-2 cardinal transfer theorems, that is, the transfer theorem

$$\langle\kappa^{++}, \kappa\rangle \rightarrow_{\mathscr{F}} \langle\lambda^{++}, \lambda\rangle$$

there are also several results. In [Vi∞], it is proved that

$$\langle\kappa^{++}, \kappa\rangle \rightarrow_{\mathscr{F}} \langle\lambda^{++}, \lambda\rangle$$

for λ regular, $\lambda < \kappa$ under $V = L$. Furthermore, if the original structure $\mathfrak{A}$ of type $\langle\kappa^+, \kappa\rangle$ omits a family of types, then the final structure $\mathfrak{B}$ of type $\langle\lambda^{++}, \lambda\rangle$ also omits the family of types.

3. We have assumed that the language $\mathscr{L}$ in $L_{\kappa\omega}(\mathscr{L})$, or its corresponding fragment, is of size at most κ. It is possible to provide an $\mathscr{F}$-consistency property for $L_{\kappa\omega}(\mathscr{L})$ for fragments $\mathscr{F}$ of size $> \kappa$, and to demonstrate the existence theorem.

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