COMPACTNESS PHENOMENA AROUND THE CLASS OF LOCALLY PROJECTIVE MODULES

JUAN ANTONIO NIDO VALENCIA
HÉCTOR GABRIEL SALAZAR PEDROZA
LUIS MIGUEL VILLEGAS SILVA

ABSTRACT. In this paper we examine compactness phenomena in some classes of modules. We first demonstrate that a κ-projective module M is projective when the size κ of M is weakly compact. We introduce the class of κ-locally projective modules and prove that they have the compactness property for R when κ is a singular or a weakly compact cardinal and R is a PID. In the case when κ is singular, we show that Shelah's Singular Compactness Theorem holds for these modules and also for torsionless modules. We show that for some not weakly compact cardinal κ, compactness does not hold for locally projective modules. Finally, we prove some compactness properties for U-torsionless modules, where U is a bimodule, for certain special classes of large cardinals κ.

1. INTRODUCTION

The study of κ-free R-modules, that is, R-modules having the property that "most submodules" generated by $< \kappa$ elements are free, has been mainly focused on determining which pairs consisting of a cardinal κ and a ring R present the *compactness property*. By this, we mean to determine if every $\leq \kappa$-generated κ-free R-module is free (in which case we also say that κ *has the compactness property for* R). What "most submodules" means in this definition depends on the kind of ring R we are dealing with. In the case of κ-free abelian groups, "most submodules" simply means "all subgroups". However, for modules over arbitrary rings R, one cannot expect all submodules to be free, so "most submodules" will stand for a specific family of free submodules, with certain closure properties (see Chapter 4 of [7] and [20, Section 6, p. 70]).

In this paper, we use these ideas to introduce the class of κ-locally projective modules, which, by analogy, are those modules having the property that "most submodules" generated by $< \kappa$ elements are locally projective.

Locally projective modules constitute a subclass of the *flat Mittag-Leffler* modules which were introduced by Raynaud and Gruson [11] as a means to prove that projectivity is a local property for the fpqc (fidelment plat et quasi compact) topology. These modules were introduced in [11] as *flat strict Mittag-Leffler* modules and lie intermediate between projective and flat modules. Recently [12] this and other related classes of modules have deserved special attention.

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This paper is structured as follows. In Section 2 we give the definition of a locally projective module due to Zimmermann-Huisgen [23] and show some connections of this class of modules with pure and torsionless modules.

We have our first approach with compactness for a singular cardinal λ in section 3. We show by means of Shelah's Singular Compactness Theorem that, both, λ-torsionless modules and λ-locally projective modules satisfy Shelah's Theorem conditions. That is, λ-torsionless modules are torsionless and λ-locally projective modules are locally projective when λ es a singular cardinal such that $\lambda > |R|$.

In section 4 we prove that κ-"free" modules of cardinality κ are "free", where "free" means projective or locally projective (in this case R is a PID) and also prove the corresponding compactness property for torsionless modules.

In Section 5 we provide a counterexample to show that compactness is not a trivial property for locally projective modules. We build a κ-locally projective module L for a cardinal κ which is not weakly compact, such that L is not locally projective. In order to do this we developed several notions and ideas to adapt an example of Wald [22] to κ-locally projectiveness. This also proves that "small" large cardinals (the least strongly inaccessible) do not have the compactness property.

Finally, in section 6 we examine compactness for U-torsionless modules, a natural generalization of torsionless, where U is an (R, S)-bimodule. In the case of U-torsionless modules we face some compatibility problems between the size κ of these modules and the size of U which may be of size larger than κ. So, in order to circumvent these issues we are led to consider special classes of large cardinals κ like measurable cardinals or the strongly unfoldable cardinals.

Throughout this work we will let $M = M_R$ be a right R-module, where R is an infinite associative ring with 1.

2. Background on Locally Projective Modules

Locally projective modules appear under different names like *flat strict Mittag-Leffler* [11], *universally torsionless* [9], *trace R-modules* [17]. It was Zimmermann-Huisgen [23] who called them *locally projective*.

Definition 2.1. M is called *locally projective* if for each epimorphism $\varphi : A \twoheadrightarrow B$, each homomorphism $\gamma : M \longrightarrow B$ and each finitely generated submodule F of M, there is a homomorphism $\gamma' : M \longrightarrow A$ such that $\gamma \restriction F = \varphi(\gamma' \restriction F)$.

$$\begin{tikzcd}
 & F \arrow[d, hook] \arrow[ddl, dashed, bend right, "\gamma' \restriction F"'] \arrow[ddr, bend left, "\gamma \restriction F"] & \\
 & M \arrow[dl, dashed, "\gamma'"'] \arrow[dr, "\gamma"] & \\
A \arrow[rr, two heads, bend right, "\varphi"'] & & B
\end{tikzcd}$$

It is clear that the class of projective modules satisfies Definition 2.1 and a large list of examples of locally projective modules that are not projective can be obtained from [23, 2.3], so the class of projective modules is a proper subclass of that of locally projective modules.

We recall the notions of *pure submodule* and *pure closure*.

As above, we show that $T \models |\overline{M}| < \kappa$. We deduce that

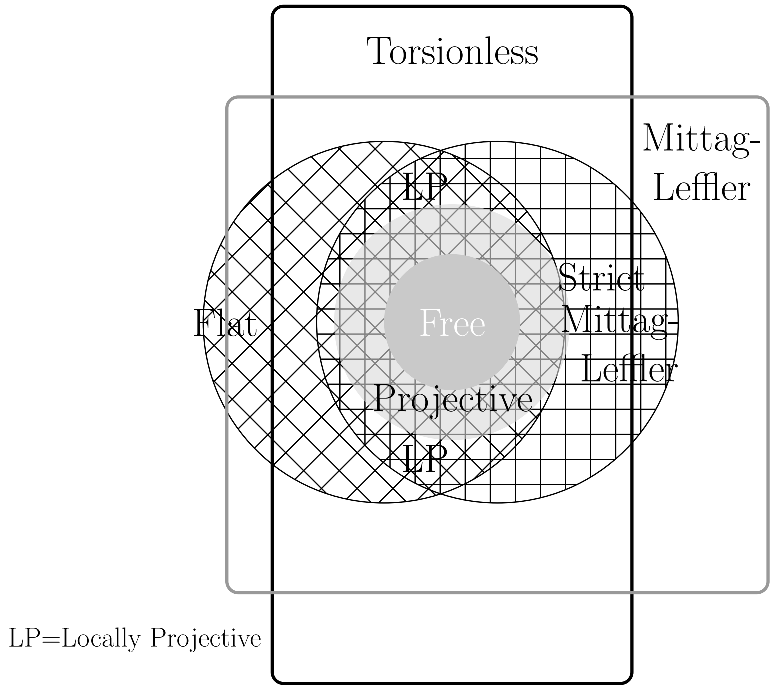
$$T \models \forall\, \overline{m}\, (\overline{m} \neq 0 \to \exists\, f(\mathrm{Hom}(f, \overline{M}, j(U), j(R)) \wedge f(\overline{m}) \neq 0)).$$

If M were not U-torsionless, we can pick $m \in M$, $m \neq 0$, such that for any f, if $\mathrm{Hom}(f, M, U, R)$, then $f(m) = 0$. We express this in T by means of $\overline{m} = j(m)$ to conclude

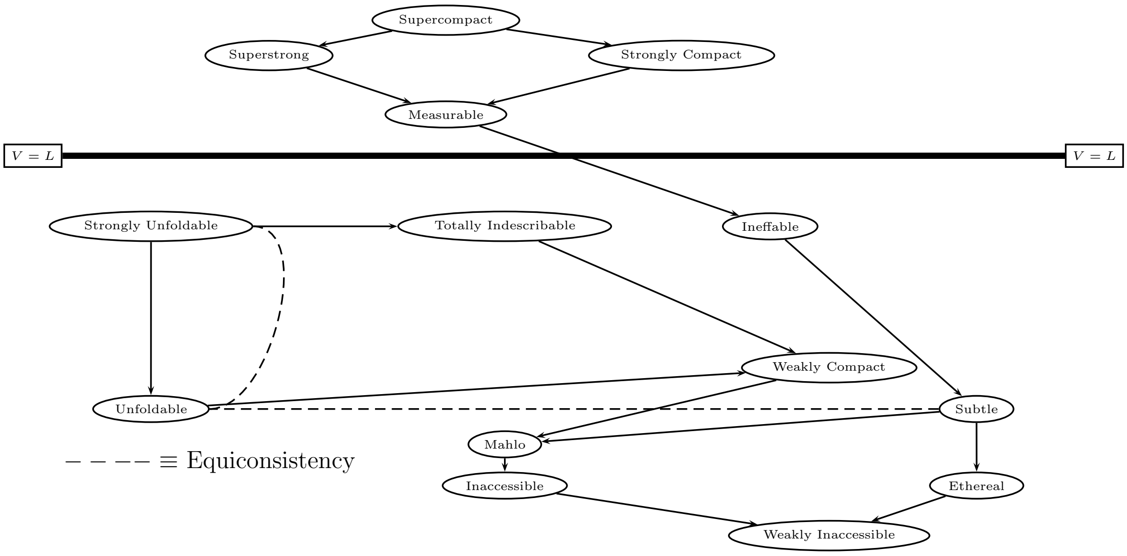
$$T \models \forall\, f(\mathrm{Hom}(f, \overline{M}, j(U), j(R)) \to f(\overline{m}) = 0),$$

which is a contradiction. ❐

Remark 6.11. In [20, p.70] Rothmaler introduced the notion of κ-Mittag-Leffler module. There, it is also noticed that κ-Mittag-Leffler modules are Mittag-Leffler for any infinite cardinal κ. In fact, $\aleph_1$-projective is the same as Mittag-Leffler.



Large Cardinals



The arrows indicate direct implications or relative consistency.

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[Nido] Maestría en Ciencias de la Complejidad, Universidad Autónoma de la Ciudad de México, San Lorenzo 290, Col. Del Valle Sur, C.P. 03100, Mexico City, Mexico.
[Salazar] Departamento de Ingeniería en Minas, Metalurgia y Geología, Universidad de Guanajuato, Ex Hacienda de San Matías S/N, Col. San Javier, C.P. 36020, Guanajuato, Gto., Mexico.
[Villegas] Departamento de Matemáticas, Universidad Autónoma Metropolitana Iztapalapa, Avenida San Rafael Atlixco 186, Col. Vicentina, C.P. 09340, Mexico City, Mexico.
e-mail: nido.juan@gmail.com, hg.salazar@ugto.mx, lmvs@xanum.uam.mx