NON REGULAR ULTRAFILTERS AND LARGE CARDINALS

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ABSTRACT. We show that, under $2^{\underset{\smile}{\kappa}} = \kappa$, the existence of a weakly normal ultrafilter on a cardinal κ implies that κ is superineffable (in particular, completely ineffable) in L. To this end we show, under the same hypothesis, that some kinds of non regularity in ultrafilters produce superineffable cardinals in L, under the above hypothesis. This extends results of Jensen and Ketonen. We provide similar conclusions for n-ineffable cardinals.

1. INTRODUCTION

The notion of a (μ,κ)-regular ultrafilter was introduced in [Ke64] and it was used to determine the cardinality of ultraproducts modulo such ultrafilters. If we use a regular ultrafilter, the cardinality of the ultrapower is the biggest possible. That is, if $\mathfrak{A}$ is an $\mathscr{L}$-structure of size $\lambda \geq \omega$ and $\mathscr{U}$ is an uniform regular ultrafilter on κ, then $\prod \mathfrak{A}/\mathscr{U}$ has cardinality λ^κ. When working with abstract logics, the authors have come across an unexpected relationship between compactness of some logics and (λ,μ)-regular ultrafilters (see [Ma85, Theorem 1.4.4, p. 655]). A natural question is, of course, does there exist a relationship between (non) regular ultrafilters and large cardinals?

The problem of whether every ultrafilter is regular, was also posed by Keisler (see [Ke72]). In [Pr70] it is shown that, under $V = L$, any uniform ultrafilter on ω_1 is regular, Thereafter, Jensen [Je] extracted a combinatorial principle PH_{ω_1} which implies Prikry's result. Then, Jensen generalised this principle to $PH_{\kappa\lambda}$ in order to prove that, if $\mathscr{U}$ is an uniform ultrafilter on κ which is γ^+-regular ($\aleph_0 \leq \gamma \leq \kappa$) and $PH_{\kappa\gamma^+}$ holds, $\mathscr{U}$ is γ-regular. Finally, Donder [Do88] showed that under $V = L$, any uniform ultrafilter on an infinite cardinal κ is regular.

In [La82] and [FoMaShe88] the authors, assuming that there exists an ω_1-dense uniform ideal on ω_1, obtained uniform ultrafilters $\mathscr{U}$ in $\aleph_1$ such that $|\omega^{\aleph_1}/\mathscr{U}| = \aleph_1$, under CH or $\Diamond_{\omega_1}$. Then, Huberich [Hu94] detached such cardinal arithmetical assumption to conclude $|\omega^{\omega_1}/\mathscr{U}| = 2^{\aleph_0}$.

In [Ke76] Ketonen concludes that $\neg 0^\#$ implies every uniform ultrafilter on κ^+ is (κ,κ^+)-regular. Then in [DoJeKo81] the authors weakened the assumption to $\neg L^\mu$ (there is no inner model with a measurable cardinal).

On the other hand, in [BeKe74] it is shown that if there is a non-(κ,κ^+)-regular ultrafilter on κ^+, then κ^{++} is inaccessible in L. In [Ke76] Ketonen conclude that, if $\mathscr{U}$ is a weakly normal ultrafilter on κ and $\mathscr{U}$ is non-(γ,κ)-regular for any $\gamma < \kappa$, then κ is Π^1_n-indescribable. In [Ka76, p. 398] it is mentioned that the following result is proved in [JeKo]. If κ is regular, $2^{\underset{\smile}{\kappa}} = \kappa$, and there is a uniform weakly normal ultrafilter on κ, then $0^\#$ exists andκ is ineffable in L.

In this paper we strengthen the last result by showing that such κ is actually superineffable and n-ineffable. Recall that a cardinal κ is ineffable if and only if for any sequence $\langle S_\alpha : \alpha < \kappa \rangle$ there exists a subset $S \subseteq \kappa$ such that $\{\alpha < \kappa : S_\alpha = S \cap \alpha\}$ is stationary in κ. A generalization of ineffability is inspired by M-ultrafilters. Kunen proved (see for instance [AHKZ77, Theorem

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2.1.2, p. 40]) that, there exists an M-ultrafilter on the cardinal κ in a transitive countable model of ZFC M if and only if κ is weakly compact according to M. Also in [AHKZ77, Theorem 2.2, p. 41], it is shown that in M, there exists a normal M-ultrafilter if and only if κ is completely ineffable. So, completely ineffability is a natural generalization of ineffability, in turn, any superineffable cardinal is completely ineffable. In other direction, Baumgartner introduced (see [Bau73]) the n-ineffable cardinals, by extending the definition of ineffable in terms of coherent families of sets (see section 7 for a detailed description of this class of large cardinals).

We would like to stress that most of our results *are unimaginative adaptations of deep imaginative methods* of R. Jensen. We consider this work as a starting point in the search for analogous results with new classes of large cardinals or in higher inner models (see open questions at the end of the paper).

2. Preliminaries

We use Gothic letters $\mathfrak{A}$, $\mathfrak{N}$, $\mathfrak{M}$, ... to denote structures and the corresponding roman letters $A, N, M, \ldots$ to represent their universes. When $f : A \longrightarrow B$ is a map, if $x \subseteq A$, $f[x]$ means$\{f(y) : y \in x\}$, and for $b \subseteq B$, $f^{-1}[b] = \{a \in A : f(a) \in b\}$. For a set A, and a cardinal α, $[A]^\alpha$ denotes the set of all subsets of A of size α, and $\wp(A)$ the set of all subset of A.

We shall assume familiarity with regular ultrafilters, as well as weakly normal ultrafilters. We refer the reader to [BeKe74], [Ka76] and [Ke76] for such concepts. About completely ineffable and n-ineffable cardinals the reader might consult [AHKZ77], [Bau73] and [Bau77]. We will primarily work with the Gödel hierarchy $(L_\alpha : \alpha \in On)$ which can be safely replaced by the J-hierarchy $(J_\alpha : \alpha \in On)$. A very good references to this subject is [Je∞]. As usual, if x is a set, $TC(x)$ is its transitive closure. For a cardinal λ, H_λ is the class of sets of hereditary cardinality less than λ. All our ultrafilters will be uniform on a regular cardinal κ.

We recall some properties of the Gödel pairing function $\Gamma : On \times On \longrightarrow On$.

Lemma 2.1. *(1) For every $\alpha, \beta \in On$, $\alpha, \beta \leq \Gamma(\alpha, \beta)$.*
(2) $\Gamma(\alpha, \beta) = \Gamma(0, \beta) + \alpha$ if $\alpha < \beta$.
(3) $\Gamma(\beta, \alpha) = \Gamma(0, \beta) + \beta + \alpha$ if $\alpha \leq \beta$.

Proof. See for instance [FeVi17, p. 448-449]. ❐

A very well known fact about L is the following.

Lemma 2.2 ($V = L$). *For every cardinal κ we have $H_\kappa = L_\kappa$.*

❐

3. Completely ineffable and superineffable cardinals

We now define the class of large cardinals related to our results. The cardinal κ is called ineffable if for any sequence $(S_\alpha : \alpha < \kappa)$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$ there exists $S \subseteq \kappa$ such that the set $\{\alpha < \kappa : S \cap \alpha = S_\alpha\}$ is stationary in κ. Notice that ineffability is an $\forall\exists$-version of $\Diamond_\kappa$. A set $A \subset \kappa$ is called ineffable if for any sequence $\langle S_\alpha : \alpha \in A\rangle$, $S_\alpha \subseteq \alpha$ there is $S \subset \kappa$ with

$$\{\alpha \in A : S \cap \alpha = A_\alpha\} \text{ stationary.}$$

The set $\mathfrak{F}_{ineff} = \{A \subset \kappa : \kappa \setminus A \text{ is not ineffable}\}$ is a κ-complete, normal filter and

$$A \in \mathfrak{F}_{ineff} \text{ when and only when } C \cap A \in \mathfrak{F}_{ineff} \text{ for every club } C \subset \kappa,$$

see Baumgartner [Bau73]. Among the properties of ineffable cardinals we find the following. The cardinal κ is ineffable if and only if whenever $f : [\kappa]^2 \longrightarrow 2$, we can pick a stationary set $X \subseteq \kappa$ such that $|f[[X]^2]| = 1$. So, an ineffable cardinal is weakly compact; any ineffable cardinal κ is Π^1_2-indescribable and it favours that $\diamondsuit_\kappa$ holds. If κ is ineffable, $(\kappa$ is ineffable$)^L$ (according to L), and there are no κ-Kurepa trees (see [De84, pp. 312-317]). It is clear that the ineffability of κ is a Π^1_3-property of $\langle V_\kappa, \in \rangle$, hence the least ineffable cardinal is not Π^1_3-indescribable.

Definition 3.1. The cardinal κ is completely ineffable if and only if there is a non empty collection Q of stationary subsets of κ such that for any $X \in Q$ and $(S_\alpha : \alpha < \kappa)$ with $S_\alpha \subseteq \alpha$ for $\alpha < \kappa$, there is an S such that $\{\alpha \in X : S \cap \alpha = S_\alpha\} \in Q$.

Hence, any completely ineffable cardinal is ineffable.

Definition 3.2. Let κ be a cardinal. We call a class $\mathscr{E} \subseteq \wp(\kappa)$ a stationary class if it satisfies the following properties.

- $\mathscr{E} \neq \emptyset$.
- For any $S \in \mathscr{E}$, S is stationary in κ.
- If $S \in \mathscr{E}$ and $T \supseteq S$, then $T \in \mathscr{E}$.

It is easy to corroborate that Definition 3.1 is equivalent to the next.

Definition 3.3. We call κ completely ineffable when there exists a stationary class $\mathscr{E}$ such that for any $X \in \mathscr{E}$ and each sequence $\langle S_\alpha : \alpha < \kappa \rangle$ with $S_\alpha \subseteq \alpha$ for every $\alpha < \kappa$, we can find $S \subseteq \kappa$, $Y \in \mathscr{E}$ with $Y \subseteq X$ such that

$$Y \subseteq \{\alpha < \kappa : S \cap \alpha = S_\alpha\}.$$

Let $C', C \subseteq 2^\kappa$. The family C' is called a flip of C, in symbols $C' \sim C$, when for every $x \in C$, either $x \in C'$ or $\kappa - x \in C'$, and for each $x \in C'$, either $x \in C$ or $\kappa - x \in C$. It follows that $\sim$ is an equivalence relation. The notation ΔC indicates the diagonal intersection of the family C.

Consider the following definition of completely ineffable in terms of flips.

Definition 3.4. The cardinal κ is completely ineffable if and only if κ is regular and there exists $Q \subseteq 2^{2^\kappa}$ such that

(1) Any $\mathscr{C} \in Q$ has size $\leq \kappa$ and $\Delta \mathscr{C} \neq \emptyset$.
(2) For each $\mathscr{C} \subseteq 2^\kappa$ of size $\leq \kappa$ there is a flip $\mathscr{C}' \in Q$ of $\mathscr{C}$.
(3) For every $\mathscr{C} \in Q$ and for any $\mathscr{D} \subseteq 2^\kappa$ with $|\mathscr{D}| \leq \kappa$ we can find a flip $\mathscr{D}'$ of $\mathscr{D}$ in such a way that $\mathscr{C} \cup \mathscr{D}' \in Q$.

According to [AHKZ77, Theorem 1.4], 3.3 and 3.4 are equivalent Definitions. From Definition 3.1 we can verify that there is a Σ^2_1 description over $\langle V_\kappa, \in \rangle$ of the complete ineffability of κ, saying $\exists Q \in \wp^2(\kappa)$ satisfying a second-order formula. Also, from [Bau77] we can derive a Π^2_1-description. Hence, the least completely ineffable cardinal is neither Σ^2_1 nor Π^2_1 indescribable. On the other hand, any completely ineffable κ is Π^1_n-indescribable for all n (see [AHKZ77, Corollary 3.5.5]). About their existence in L, if κ is completely ineffable, then $(\kappa$ is completely ineffable$)^L$ (see [AHKZ77, Theorem 4.1.1]).

There is also a characterization of completely ineffable cardinals in terms of M-ultrafilters: There exists an ultrafilter U such that U is a normal M-ultrafilter on κ if and only if $M \models \kappa$ is completely ineffable (see [Kl78]).

We now present the main class of large cardinals for our results.

Definition 3.5. κ is *superineffable* if for every sequence $\langle S_\alpha : \alpha < \kappa \rangle$, $S_\alpha \subseteq \alpha$ there are $S \subset \kappa$ such that A is ineffable and

$$A \subseteq \{\alpha < \kappa : S \cap \alpha = S_\alpha\}$$

If κ is superineffable, take $\mathscr{F}_{ineff}$ as a stationary class $\mathscr{E}$ (recall that a normal filter in κ contains any club $C \subseteq \kappa$). Let $X \in \mathscr{E}$ and let $(S_\alpha : \alpha < \kappa)$ be a sequence with $S_\alpha \subseteq \alpha$ for every

(a) The α-Erdős cardinals, when α is countable according to L. We recall the arrow notation $\kappa \longrightarrow [\alpha]^{<\omega}_{\gamma}$ which means: if $f_i : [\kappa]^{n_i} \longrightarrow \gamma$, $i < \omega$, then there exists $X \subseteq \kappa$ with $otp(X) \geq \alpha$ and X is homogeneous for every f_i. That is,

$$f_i[[X]^{n_i}] = \{\nu_i\},$$

for some $\nu_i < \gamma$, $i < \omega$. When this arrow relationship holds, κ is called α-Erdős cardinal. In [Sil70] it is proved that if κ is α-Erdős cardinal in he universe and α is countable according to L, then κ is α-Erdős in L. It is well known that the existence of α-Erdős cardinals for $\alpha \geq \omega_1$ is incompatible with $V = L$.

(b) The totally indescribable cardinals.

(c) The Π^m_n-indescribable cardinals.

(d) The weakly-Ramsey Cardinals. See [Gi11].

(e) The strongly unfoldable cardinals. Villaveces [Vill98] defines the following notion of strong unfoldability of a cardinal.

Definition 8.1. A cardinal κ is θ-strongly unfoldable if for every κ-model W there exists an elementary embedding $j : W \hookrightarrow N$, where N is a transitive set, the critical point of j is κ and $V_\theta \subseteq N$. We say that κ is strongly unfoldable iff it is θ-strongly unfoldable for any ordinal θ.

Actually, he first introduces the notion of unfoldable cardinal in terms of chains of end elementary extensions of $\langle V_\kappa, \in, S \rangle$ for any $S \subseteq \kappa$, but this is easily seen to be equivalent to one in terms of elementary embeddings. While measurable cardinals are characterized by elementary embeddings whose domain is all of V, strongly unfoldable cardinals carry embeddings whose transitive domain mimics the universe V, yet is a set of size κ.

Villaveces shows the following: If κ is unfoldable in the universe, then it is unfoldable in L. If $V = L$, then strong unfoldability and unfoldability (see [Vill98]) coincide.

Strongly unfoldable cardinals strengthen weakly compact cardinals similarly to the way strong cardinals strengthen measurable cardinals. Their consistency strength is well below measurable cardinals. Strongly unfoldable cardinals also exhibit some of the characteristics of supercompact cardinals.

(4) Consider the core model for measures of order 0 (see [Je88]) K, or more general, a core model K which can contain up to one strong cardinal (see [Je2]). which classes of large cardinals above superineffable can be used to produce new results in K?

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