A κ-ROUGH MORASS UNDER $2^{<\kappa} = \kappa$ AND VARIOUS APPLICATIONS

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Abstract. Let κ be an uncountable regular cardinal. Assuming $2^{<\kappa} = \kappa$, we construct a κ-rough morass. As an immediate consequence of this result, we establish a proof of the Gap-1 cardinal transfer theorem under $2^{<\kappa} = \kappa$. We will examine how this affects the consistency strength of this transfer problem. We will also present several applications of our rough morass.

1. Introduction

It is worth noting at the outset that a rough morass $\neq$ coarse morass $=$ weak morass. Historically, the structures called morasses have been associated with proof of cardinal transfer theorems in model theory, but they are, in fact, strong combinatorial principles capable of solving involved problems in set theory, topology and model theory. However, their intricate structures prevent mathematicians from applying them, as they do with other combinatorial principles like $\diamondsuit$, $\square$, MA, etc. Morasses can be seen either as complex systems of indices or as sophisticated types of direct limits. They depend on two ordinal parameters. Indeed, we should examine a (κ, λ)-morass, where κ is a regular uncountable cardinal and λ is an ordinal not greater than κ. Their existence can be proved in L ([Dev84], [We10] both for $\lambda = 1$) or introduced by forcing ([St77]). A morass satisfies two sets of axioms ([We10, pp. 725-726]): CP1 and CP2. If we demand CP1 alone, we obtain a weak morass. This could be seen as an insignificant relaxation of the definition of a morass, because the existence of weak morasses had been only known under $V = L$-like assumptions ([Do81], [Ra05]). Nevertheless, even under these circumstances, weak morasses are more accessible tools than full morasses; they are easier to use but strong enough to prove several important combinatorial principles ([Do81]) and model theoretic results ([Vill06], [Vill10]). However, since we desist from CP2, demonstrating the existence of a weak morass in L avoids the use of deeper results from the fine structure of L. Unfortunately, until now weak morasses have been obtained only through generic extensions- or, as we mentioned, under assumption of a constructible nature, like $V = L$. In the search for similar principles under a weaker hypothesis, we found a structure which we call a rough morass that approximates a coarse morass; under $2^{<\kappa} = \kappa$ we are able to construct a κ-rough morass. A full morass is, in particular, a coarse morass and "coarse" implies a rough morass. However, these implications are not generally reversible. This paper aims to study rough morasses. As we shall see, a κ-rough morass allows us to derive some

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important combinatorial principles such as $\Box^*_\kappa$ and $\diamondsuit^\#_\kappa$ for suitable κ in a constructible-like universe, and we can also obtain proofs for some cases within the Gap-1 cardinal transfer theorem.

Let κ, λ be infinite cardinals, let $\mathcal{L}$ be a first-order language with at least one unary predicate symbol and let $\mathfrak{A}$ be an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, U^{\mathfrak{A}}, \ldots \rangle$, where $|\mathfrak{A}| = \lambda^+$, and $|U^{\mathfrak{A}}| = \lambda$. The Gap-1 cardinal transfer theorem

$$(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$$

assures us that we can find an $\mathcal{L}$-structure $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$ such that $|\mathfrak{B}| = \kappa^+$ and $|U^{\mathfrak{B}}| = \kappa$.

The cardinal transfer theorem

$$(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$$

for λ a regular cardinal and $|\mathcal{L}| \leq \lambda$ under the assumption that there exists a λ-weak morass is proved in [Vill06]. In this paper, we shall demonstrate how that proof can be modified to use a rough rather than a weak morass. This significantly augments what is known about the consistency strength of the Gap-1 cardinal transfer theorem. Furthermore, as we have already mentioned, we shall show that some combinatorial principles ($\diamondsuit^*_\kappa$, $\Box^*_\kappa$ and $\diamondsuit_{\kappa^+}$, for certain κ) are consequences of the existence of such a morass under appropriate hypotheses. The proofs for all such results employ rough morasses in different ways. We seek here to analyse those structures in detail through their applications; we hope more applications for rough morasses will be found and incorporated into everyday set-theoretical tools.

The next section introduces notation, some auxiliary results and the primitive recursive (p.r.) functions which allow us to find the correct set W in order to work inside $L_{\kappa^+}[W]$. In section 3, we display our κ-rough morass, and in section 4, we provide its applications. We always assume ZFC.

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2. Preliminaries

I use Gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ to denote structures and the corresponding Roman letters $A, B, C, \ldots$ to represent their universes. For our language $\mathcal{L}$, we say that an $\mathcal{L}$-structure $\mathfrak{D}$ is of type (λ, μ) if $\mathfrak{D} = (D, U^{\mathfrak{D}}, \ldots)$, $|D| = \lambda$ and $|U^{\mathfrak{D}}| = \mu$. By $\lim((\mathfrak{B}_i : i \in I), (\sigma_{ij} : i < j, i, j \in I))$ we mean the direct limit of the given directed system. If $f : X \longrightarrow Y$ and $A \subseteq X$, then $f[A] = \{f(a) : a \in A\}$, and $f^{-1}[B] = \{x \in X : f(x) \in B\}$ for any $B \subseteq Y$. With $\boxed{\checkmark}$ (n), we mark the end of the proof of the Claim n. If μ is a cardinal, H_μ is the class of all sets of hereditary cardinality less than μ. For a set x, $Pw(x)$ denotes its power set and $TC(x)$ its transitive closure. The theories ZF^-, ZFC^- represent ZF, respectively ZFC, without the power set axiom, where the choice's axiom is expressed as the statement that every set is isomorphic to an ordinal. Remember that ZF includes the collection schema. V is the collection of all sets and On the class of all ordinals, while $Or(x)$ means x is an ordinal. The notation $f : A \leftrightarrow B$ signifies that f is a bijection between A and B, $f : A \twoheadrightarrow B$ that f is onto and $f : A \hookrightarrow B$ that f is an elementary embedding. With $[A]^\lambda$ we represent the set of subsets of A of cardinality λ, with the corresponding meaning for $[A]^{<\lambda}$. For the transfer theorem, we work in a first-order language $\mathcal{L} = \{\in, U, \ldots\}$, where U is a unary predicate symbol, but in order to construct the morass, we work in the language of set theory $\{\in, =\}$ which is augmented by a unary predicate symbol, A. If A is a set or a proper class, $L[A]$ ($= L^A$) is the constructible universe relative to A, which from now on we denote is bounded in $\hat{\mathbb{S}}_\alpha$; therefore, $|\overline{\mathbb{S}}_\alpha| \leq |\alpha|$ for each $\alpha < \kappa$. Moreover, $\overline{\mathbb{S}}_\alpha$ is closed for $\alpha < \kappa$, and $\overline{\mathbb{S}}_\kappa$ is a club in κ^+. For $\overline{\nu}, \nu \in \bigcup_{\alpha \leq \kappa} \overline{\mathbb{S}}_\alpha$, we define a new order $\lhd_\kappa$:

$$\overline{\nu} \lhd_\kappa \nu \Leftrightarrow \overline{\nu} \lhd \nu \text{ and } F \restriction \alpha_\nu \in rng(\pi_{\overline{\nu}\nu}).$$

Lemma 4.9. *Let $\overline{\nu} \lhd \nu$, $\nu \in \overline{\mathbb{S}}_\alpha$ such that $F \restriction \alpha \in rng(\pi_{\overline{\nu}\nu})$. Then $\overline{\nu} \in \overline{\mathbb{S}}_{\alpha_{\overline{\nu}}}$, hence $\overline{\nu} \lhd_\kappa \nu$.*

Proof. Since $F \restriction \alpha \in rng(\pi_{\overline{\nu}\nu})$, we find $\overline{F} \in L_{\overline{\nu}}^{W_{\overline{\nu}}}$ such that $\pi_{\overline{\nu}\nu}(\overline{F}) = F \restriction \alpha$, and by the elementarity of $\pi_{\overline{\nu}\nu}$, $\overline{F} = F \restriction \alpha_{\overline{\nu}}$. It remains to be verified that $A_{\alpha_{\overline{\nu}}} \notin L_{\overline{\nu}}^{W_{\overline{\nu}}}$. Assume the opposite to obtain a contradiction. Since $\overline{\nu} \in \hat{\mathbb{S}}_{\alpha_{\overline{\nu}}}$, there exists a club $\overline{C} \subseteq \alpha_{\overline{\nu}}$, $\overline{C} \in L_{\overline{\nu}}^{W_{\overline{\nu}}}$, with $A_\beta \neq A_{\alpha_{\overline{\nu}}} \cap \beta$ for every $\beta \in \overline{C}$. We set $C = \pi_{\overline{\nu}\nu}(\overline{C})$, $A = \pi_{\overline{\nu}\nu}(A_{\alpha_{\overline{\nu}}})$ then $\alpha_{\overline{\nu}} \in C$ because C is a club and $A \cap \alpha_{\overline{\nu}} = A_{\alpha_{\overline{\nu}}}$. Since $\pi_{\overline{\nu}\nu}$ is elementary $A \cap \beta \neq A_\beta$ for any $\beta \in C$, a contradiction is created. □

Theorem 4.10. *Let $\kappa > \omega$ be an uncountable regular limit cardinal that is not ineffable. Then, $\diamondsuit_\kappa^\#$ holds.*

Proof. We proceed as in the proof of Lemma 4.8 but with the tree order $\lhd_\kappa$ instead of $\lhd$. Define N_α for every $\alpha \in \overline{\mathbb{S}}^0$, as we did there. □

Corollary 4.11. *If κ is not ineffable, there exists a κ-Kurepa tree without κ-Aronszajn subtrees.*

Proof. In [Dev82, p.897, Theorem 3] Devlin proves the affirmation using a $\diamondsuit_\kappa^\#$-sequence. □

Corollary 4.12. *κ is ineffable if and only if $\diamondsuit_\kappa^\#$ is false.*

□

Remark 4.13. The following fact provides a feature of $L[W]$. In [AgHeVi18], it is proved that for any $A \subseteq \mu$ with μ as an uncountable cardinal and under $V = L[A]$, the structure L_μ^A is a Jónsson algebra. Under $V = L[W]$, $L_{\kappa^+}[W]$ is also a Jónsson algebra. Furthermore, there is no $\eta > \kappa^+$ such that η is a Jónsson cardinal.

5. Open Problems

(1) Let A be a set or a proper class. Let κ be a regular cardinal. Can we construct a κ-rough morass in L^A?
(2) Let A be a set or a proper class. Let κ be an uncountable regular cardinal. Assume that J_ν^A is acceptable for any $\nu \leq \kappa^+$. Under $V = J^A$, is there a κ-rough morass at κ?
(3) Let K be the core model for measures of order 0 (see [JeZe00]). Assume $V = K$. Can the existence of a κ-rough morass for any uncountable regular cardinal κ be shown?.

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