“FREE” MODULES AND VERY LARGE CARDINALS

JUAN ANTONIO NIDO VALENCIA
HÉCTOR GABRIEL SALAZAR PEDROZA
LUIS MIGUEL VILLEGAS SILVA

[Nido] Maestría en Ciencias de la Complejidad, Universidad Autónoma de la Ciudad de México, Mexico, [Salazar] Departamento de Ingeniería en Minas, Metalurgia y Geología, División de Ingenierías, Campus Guanajuato, Mexico, [Villegas] Departamento de Matemáticas, Universidad Autónoma Metropolitana Iztapalapa, Mexico

e-mail: antonio.nido@uacm.edu.mx, hg.salazar@ugto.mx, lmvs@xanum.uam.mx

ABSTRACT. We prove that for R-modules, κ-“free” implies “free” when κ is a strongly compact or a strong cardinal, and “free” means projective, locally projective or U-torsionless. When κ is strongly compact, κ-free implies free for modules over arbitrary rings. When κ is strong, κ-Baer implies Baer for modules over tame hereditary algebras. Also for κ strong, κ-free implies free for abelian groups, and as a consequence we provide an $L_{\kappa^+\kappa}$-sentence in the language of the theory of groups to express when an abelian group is free.

1. INTRODUCTION

An abelian group G is called κ-free if every subgroup of G of size $< \kappa$ is free. If we attempt to extend this definition to R-modules, where R is an arbitrary ring, we can face a serious difficulty, like not having any free proper submodules: when the cardinality of R is $\geq \kappa$ or because of the absence of the Schreier property. One overcomes this obstacle by giving a suitable definition of κ-free. In general, we can define κ-“free” R-modules, where “free” can be some suitable class of modules other than that of free modules. The research on κ-“free” R-modules, that is, R-modules having the property that most submodules generated by $< \kappa$ elements are “free”, has proved to be a very fruitful approach using set theoretical tools to study some classes of modules. In particular, we are concerned with determining when a κ-“free” module is, in fact, “free”. In the case of κ-free abelian groups, “most submodules” means all subgroups of size $< \kappa$. In the general case of R-modules, “most submodules” will stand for a specific family of “free” submodules of cardinality $< \kappa$, with certain closure properties (see Chapter 4 of [EkMe02]). We will consider the classes of torsionless, U-torsionless, free, Baer, projective and locally projective modules as “free”.

This article serves two fundamental purposes. To examine such classes of R-modules from the very large cardinals perspective and to examine these large cardinals from the perspective of those classes of modules. The large cardinals we will consider are the strongly compact and the strong

---

2020 *Mathematics Subject Classification.* Primary 03C20, 03C55, 03C60, 03C75; secondary 20A15, 20K99, 16D99.

Key words and phrases. Strongly Compact Cardinals, Strong Cardinals, Infinitary Logic, Definability, Free Modules, Locally Projective Modules, Projective Modules, U-Torsionless Modules, Baer Modules, Extenders, Tame Hereditary Algebras.

This work was partially supported (Villegas) by SECIHTI (SNII).

ones. Both of them are in the upper level of the large cardinals hierarchy. A supercompact cardinal is strongly compact and strong, but there is not a known relationship between strongly compact and strong. For instance, strongly compactness does not imply strongness since the least measurable cardinal can be strongly compact and strong implies measurable, but the first measurable κ is not even $(\kappa+2)$-strong. The order given by "belonging to the class A implies belonging to the class B" becomes highly unpredictable at this level of the hierarchy.

We provide some known relationships (see [Go22], [Je08] and [Ka09]) between "similar" cardinals in terms of their largeness: superstrong, strong, Woodin and strongly compact. If there exist superstrong and strongly compact cardinals, the least superstrong is less than the least strongly compact. If κ is superstrong, κ is Woodin. In fact, there are κ Woodin cardinals below κ. Every supercompact cardinal is Woodin and below a Woodin cardinal δ, we find δ cardinals that are λ-strong for each $\lambda < \delta$.

Another way to compare classes of large cardinals is through consistency strength. A class A of large cardinals is consistencywise stronger than a class B ($A \geq B$) if the consistency of the system ZF+A implies the consistency of ZF+B. These are relative consistencies over the system ZF. By definition, strong cardinals lie below supercompact cardinals and above measurable cardinals in the consistency strength hierarchy. Concerning the consistency strength of the strongly compact cardinals, let us consider the following: it is proved in [St05] that if $\square_\kappa$ fails and κ is a strong limit singular cardinal then AD holds in $L(\mathbb{R})$. Solovay showed that if κ is strongly compact, $\square_\mu$ fails for large enough μ so that strongly compact implies $AD^{L(\mathbb{R})}$. In [La01, Lemma 2.5.21] it is shown that if κ is strongly compact, there are many $\gamma < \kappa$ such that if G is V-generic for the full stationary tower $\mathbb{P}_{<\delta}$, then there is an elementary embedding $j : V \hookrightarrow W$ in $V[G]$ such that $\mathrm{crit}(j) < \gamma$ and $j(\mathrm{crit}(j)) < \gamma$ and $M^{<\gamma} \cap V[G] \subseteq M$. [St96, Theorem 7.1] assures that if a given forcing of size less than a measurable κ adds such an elementary embedding, then V_κ has an inner model with a Woodin cardinal. This short story serves to illustrate the complex interrelationships between these classes of large cardinals. However, there is not a clear evidence about a relationship between the consistency strength of strong and strongly compact cardinals.

A relevant problem in the model theory of modules is that of the definability of classes of modules in the first order logic $L_{\omega\omega}$ in the language of the theory of modules. For instance, the class of the torsion R-modules is not definable in that logic and, also, many other classes of modules are not definable. That prevents us from using the tools from model theory to study those R-modules. However, the mere fact of knowing the existence of a set of sentences that define a class, not even knowing exactly what those sentences are, offers great benefits to the use of model theory. The classes that are not $L_{\omega\omega}$-definable can be definable in an infinitary logic $L_{\lambda\kappa}$ or in some other abstract logics.

At the end of this paper we will show that for abelian groups and κ strong, κ-free implies free. This fact allows for the definability of the class of free abelian groups in $L_{\infty\kappa}$ ([Me76] obtained a similar result for κ strongly compact). We obtained that κ-"free" implies "free" for other classes of R-modules when κ is a strongly compact or a strong cardinal.

Moreover, the specific proofs throw light on the similarities and differences among those classes of large cardinals. So, for instance, the use of ultrapowers given by extenders (for strong cardinals) allows us to recreate to a certain extent the compactness of the strongly compact cardinals. The use of infinitary languages (for strongly compact cardinals) provides us of information about other classes of modules that could be definable in some abstract logic.

We can now summarize the content of this work. When κ is a strong or a strongly compact cardinal, we prove that κ-"free" implies "free", where "free" means locally projective and U-torsionless

(torsionless). When "free" means projective, we prove that the direct limit of a κ-directed system of projective modules is projective, from which it follows that κ-projectiveness implies projectiveness. When κ is strongly compact, we prove that all κ-free modules of any size are free, whereas for strong cardinals we still are unable to prove this result. This is a remarkable difference in our context between strong and strongly compact cardinals. When κ is strong and R is a tame hereditary algebra, we prove that a κ-Baer R-module is Baer. Finally, it is known that κ-free implies free when κ is a strongly compact cardinal. We prove the same result for κ strong. As a corollary, we show the $L_{\kappa^+\kappa}$-definability of being a free abelian group.

If κ is a weakly compact cardinal, κ-free, κ-locally projective, κ-U-torsionless and κ-projective modules are studied in [EkMe02], [FuSa01] and [NiSaVi20], where it is proved that a κ-"free" module of size κ is "free".

Throughout this paper R denotes a ring, and all modules, unless otherwise stated, will be left R-modules. Whenever we require M to be torsion-free, we will assume R is a Noetherian integral domain.

About notation: if $f : X \longrightarrow Y$, then for any $A \subseteq X$ and $B \subseteq Y$ let $f[A] = \{f(a) : a \in A\}$ and $f^{-1}[B] = \{x \in X : f(x) \in B\}$. The theories ZF^-, ZFC^- represent ZF, respectively ZFC, without the power set axiom, where the Axiom of Choice is expressed as the statement: for every set x there exist an ordinal γ and a bijection $b : x \longrightarrow \gamma$. The notation $f : A \leftrightarrow B$ means that f is a bijection between A and B, and $f : A \twoheadrightarrow B$ that f is onto.

We shall use the following notions in the rest of the paper (see [HeTr12]).

Definition 1.1. Let M be an R-module and κ be an uncountable regular cardinal. A directed system S of R-submodules of M is a *κ-dense system in M* if

(i) every subset of M of cardinality $< \kappa$ is contained in an element of S; and
(ii) S is closed under unions of well-ordered chains of length $< \kappa$.

Definition 1.2. (i) We say that an R-module M is *$< \kappa$-generated* (resp. $\leq \kappa$-generated) if it has a generating set of cardinality $< \kappa$ (resp. $\leq \kappa$). A *κ-generated* module is similarly defined.
(ii) Let $\mathfrak{C}$ be a class of R-modules. If κ is an uncountable regular cardinal, we say an R-module M is a *$(\kappa, \mathfrak{C})$-module*, if there is a κ-dense system $S \subset \mathfrak{C}$ in M consisting of $< \kappa$-generated modules.
(iii) An R-module M is *κ-"free"* if there is a class $\mathfrak{C}$ of R-modules such that M is a $(\kappa, \mathfrak{C})$-module.

Definition 1.3. A κ-directed set $(I, \leq)$ is a poset in which every subset of cardinality less than κ has an upper bound, that is, if $J \subseteq I$ with $|J| < \kappa$, then there exists $k \in I$ such that $j \leq k$ for every $j \in J$.

Definition 1.4. Let $(I, \leq)$ be a κ-directed set, $\{M_i : i \in I\}$ a family of R-modules and $\{\sigma_{ij} : i, j \in I, i \leq j\}$ a family of R-homomorphisms $\sigma_{ij} : M_i \longrightarrow M_j$. We say that (M_i, σ_{ij}) is a κ-directed system of R-modules over I if

(1) each $\sigma_{ij} : M_i \longrightarrow M_j$ is a monomorphism for $i \leq j$.
(2) $\sigma_{ik} = \sigma_{jk} \circ \sigma_{ij}$ if $i \leq j \leq k$.

Definition 1.5. Let $D = (M_i, \sigma_{ij})$ be a κ-directed system of R-modules over I. The direct limit of the system D is an R-module M together with R-monomorphisms $\sigma_i : M_i \longrightarrow M$ such that $\sigma_i = \sigma_j \circ \sigma_{ij}$ if $i \leq j$ and for every $m \in M$ there are $i \in I$ and $m_i \in M_i$ such that $\sigma_i(m_i) = m$.

For a proof of the existence of the direct limit see [Ho93] 50-52.

We have dealt with the case of abelian groups. The case of R-modules requires the notion of a κ-dense system. The treatment of these systems demands novel techniques and results, which will appear in a forthcoming paper.

4. Open Problems and Remarks

1. Is it possible to prove Corollary 2.20 for strong cardinals and arbitrary infinite rings? Or even for arbitrary domains?

2. Assume that κ is a measurable cardinal and that $\mu > \kappa$, but μ is not weakly compact. Let M be a κ-locally projective (κ-torsionless) R-module of size μ, where $|R| < \kappa$. Is M locally projective (torsionless)? What happens if μ is a limit of weakly compact cardinals?

References

[AnBaHe08] Angeleri-Hügel, L., Bazzoni, S., Herbera, D., *A solution to the Baer splitting problem*, Trans. Am. Math. Soc. **360** (2008), pp. 2409-2421.

[AnHeTr10] Angeleri-Hügel, L., Herbera, D., Trlifaj, J., *Baer Modules and Mittag-Leffler Modules over tame hereditary algebras*, Matematische Zeitschrift **265** (2010), pp. 1-19.

[Ap01] Apter, A., *Supercompactness and measurable limits of strong cardinals*, J. Symbolic Logic **66**(2001), 629-638.

[Ba36] Baer, R., *The subgroup of the elements of finite order of an abelian group*, Ann. Math. **37** (1936), 766-781.

[BeLi80] Beller, A., Litman, A., *A strengthening of Jensen's $\square$ principles*, J. Symb. Logic **45** (1980), pp. 251-264.

[Da94] Dauns, J., *Modules and Rings*, Cambridge University Press, 1994.

[Ek75] Eklof, P.C., *On the existence of κ-free abelian groups*, Proc. Am. Math. Soc. **47** (1975), pp. 65-72.

[EkFuSh90] Eklof, P.C., Fuchs, L., Shelah, S., *Baer Modules Over Domains*, Trans. Am. Math. Soc. **322** (1990), pp. 547-560.

[EkMe02] Eklof, P.C., Mekler, A., *Almost Free Modules*. Amsterdam: North-Holland, 2002.

[FuSa01] Fuchs, L., Salce, L., *Modules over Non-Noetherian Domains*, American Mathematical Society, 2001.

[Go22] Goldberg, G., *The ultrapower axiom*, De Gruytier, Berlin, 2022.

[Gr73] Gregory, J., *Abelian groups infinitarily equivalent to free ones*, Notices Am. Math. Soc. **20** (1973), A-500 Abstract #73T-E77.

[Gr69] Griffith, P., *A Solution to the Splitting Mixed Group Problem of Baer*, Trans. Am. Math. Soc. **139** (1969), pp. 261-269.

[HeTr12] Herbera, D., Trlifaj J., *Almost free modules and Mittag-Leffler conditions*, Adv. Math. **229**(2012), 3436-3467.

[Ho93] Hodges, W., *Model Theory*, Cambridge University Press, Cambridge, UK, 1993.

[Je08] Jech, T., *Set Theory*, Third Milenium Ed., Springer-Verlag, 2008.

[Ka09] Kanamori, A., *The Higher Infinite*, Springer-Verlag, 2nd Ed., N. York, USA, 2009.

[La99] Lam, T. Y., *Lectures on Modules and Rings*, Springer-Verlag, 1999.

[La01] Larson, P., *The Stationary Tower*, AMS, University Lecture Series, 2001.

[Lu93] Lukas, F., *A class of infinite rank modules over tame hereditary algebras*, J. Algebra **158**(1993), 18-30.

[Ma76] Magidor, M., *How large is the first strongly compact cardinal? or a study on identity crises*, An. Math. Logic **10**(1976), 33-57.

[Me76] Mekler, A., *Applications of Logic to Group Theory*, Ph.D. Dissertation, Stanford University, 1976.

[NiSaVi20] Nido Valencia, J. A., Salazar Pedroza, H. G., Villegas Silva, L. M., *Compactness Phenomena Around the Class of Locally Projective Modules*, Houston J Math 46-4(2020), 859-881.

[Ok82] Okoh, F., *Baer Modules*, J. Algebra **77**(1982), 402-410.

[Ok94] Okoh, F., *Bouquets of Baer Modules*, J. Pure Appl. Algebra **93**(1994), 297-310.

[Pr09] Prest, M., *Purity, Spectra and Localisation*, Encyclopedia of Mathematics and its Applications. Vol. 121. Cambridge: Cambridge University Press, 2009.

[Ri79] Ringel, C.M., *Infinite dimensional representations of finite dimensional hereditary algebras*, Symposia Math. **23**(1979), 321-412.

[SaTr20] Šaroch, J., Trlifaj, J., *Test sets for factorization properties of modules*. Rend. Sem. Mat. Univ. Padova 144 (2020), pp. 217-238.

[Sc14] Schindler, R., *Set Theory*, Spinger-Verlag, Switzerland, 2014.

[St96] Steel, J., *The Core Model Iterability Problems*, Lecture Notes in Logic Vol.8, Association Symb. Logic, 1996.
[St05] Steel, J., *PFA implies* $AD^{L(\mathbb{R})}$, J. Symbolic Logic 70 (2005), pp. 1255-1296.
[Ze10] Zeman, M., *Global square sequences in extender models*, Ann. Pure App. Logic Vol.161 (2010), 956-985.