“FREE” MODULES AND SOME LARGE CARDINALS

JUAN ANTONIO NIDO VALENCIA
HÉCTOR GABRIEL SALAZAR PEDROZA
LUIS MIGUEL VILLEGAS SILVA

ABSTRACT. This paper deals with “freeness” properties for some classes of modules, relative to large cardinals. We study large and small R-modules. For large modules, we consider very large cardinals, and for small modules, we use subtle cardinals together with $V = L$.

1. INTRODUCTION

An abelian group G is called κ-free if every subgroup of G of size $< \kappa$ is free. If we attempt to extend this definition to R-modules, where R is an arbitrary ring, we can face a serious difficulty, like not having any free proper submodules. For instance, when the cardinality of R is $\geq \kappa$ or because of the absence of the Schreier property. We overcome this obstacle by giving a suitable definition of κ-free. In general, we can define κ-“free” R-modules, where “free” can be any common class of modules other than free modules. The research on κ-“free” R-modules, that is, R-modules having the property that most submodules generated by $< \kappa$ elements are “free”, has proved to be a very fruitful set-theoretical tool to study some classes of modules. In particular, we are concerned with determining when a κ-“free” module is, in fact, “free”. When this happens, we call it the “freeness” property. What “most submodules” means in this definition depends on the kind of ring R we are dealing with. In the case of κ-free abelian groups, it simply means “all subgroups”. However, for modules over arbitrary rings, “most submodules” will stand for a specific family of “free” submodules, with certain closure properties (see Chapter 4 of [7]). We will consider the classes of torsionless, U-torsionless, free, projective and locally projective modules as “free”.

We study two pairs of classes of large cardinals, namely, strong and strongly compact, and weakly compact and subtle cardinals. The first pair are located on the top of the diagram of large cardinals (which can be found on page 24), while the second pair can be found on the bottom part. Weakly compact and strongly compact cardinals yield certain degree of compactness in infinitary logic. Moreover, numerous equivalent formulations in terms of ultrafilters or combinatorial formulations stem from this fact. Subtle cardinal, in turn, recreate the diamond principle and have no evident relation with the weakly compact cardinals, except for the fact that it is known that the least subtle cardinal cannot be weakly compact. The absence of compactness in one of those classes forced us to develop radically different methods to work on them separately, namely, we must build

2010 *Mathematics Subject Classification.* Primary 16D80 16B70 03E45 03E55; secondary 03C20 03C55 03C60.

Key words and phrases. Locally projective modules, subtle cardinals, strongly compact cardinals, free modules, projective modules, compactness.

Some results of this paper were obtained while the third author was visiting The Albert Ludwig University, Freiburg, Germany. He would like to thank its mathematical logic department for the hospitality during his stay, and he is also grateful to Heike Mildenberger and Jörg Flum for making this visit possible. This work was partially supported by a CONACYT grant 140 186412 (Estancias sabáticas).

a rather sophisticated contraption under the hypothesis $V = L$ in order to work with subtle cardinals. κ-free and κ-projective modules for κ a weakly compact cardinal were studied in [7] and [9], where it is shown that if κ is weakly compact, then a κ-"free" module of size κ is "free", when κ is not weakly compact and the ring is not left perfect there is a counterexample. We instead consider subtle cardinals and left perfect rings to carry out this construction.
This setting allowed us to establish results for subtle cardinals, which are impossible to obtain for weakly compact cardinals by changing the class of rings at issue. Strong cardinals allow an interesting characterization in terms of extenders and neither present a known relation with the strongly compact cardinals. Through the use of extenders we manage to recreate the results obtained for strongly compact cardinals. We have divided this article according to the "size" of modules: large ones correspond to strongly compact or strong cardinalities, while small ones correspond to subtle cardinalities.

We study κ-"free" for different large cardinals κ, namely, strongly compact, strong and subtle cardinals. In particular, we continue to study the class of κ-locally projective modules, which were introduced in [20]. We prove that for a strongly compact cardinal κ, κ-locally projective modules are locally projective.

Then we turn to small modules. We succeed in proving the "freeness" property for subtle cardinals κ under $V = L$ for several classes of modules: κ-locally projective, κ-free, κ-torsionless and κ-projective modules. To this end, we build a so called κ-mire which is kind of like a morass, but weaker.

In Section 2 we tackle the case of modules of large cardinality. In Section 4, we construct in L a κ-mire which will be used in Section 5 to confirm the "freeness" property of certain small modules.

Throughout this paper R denotes a ring, and all modules, unless otherwise stated, will be left R-modules. Whenever we require M to be torsion-free, we will assume R is a Noetherian integral domain.

About notation: if $f : X \longrightarrow Y$, then for any $A \subseteq X$ and $B \subseteq Y$ let $f[A] = \{f(a) : a \in A\}$ and $f^{-1}[B] = \{x \in X : f(x) \in B\}$. The theories ZF^-, ZFC^- represent ZF, respectively ZFC, without the power set axiom, where the Axiom of Choice is expressed as the statement: for every set x there exist an ordinal γ and a bijection $b : x \longrightarrow \gamma$. The notation $f : A \leftrightarrow B$ means that f is a bijection between A and B, and $f : A \twoheadrightarrow B$ that f is onto.

We shall use the following notions in the rest of the paper (see [10]).

Definition 1.1. Let M be an R-module and κ be an uncountable regular cardinal. A directed system S of R-submodules of M is a *κ-dense system in M* if

(i) every subset of M of cardinality $< \kappa$ is contained in an element of S; and
(ii) S is closed under unions of well-ordered chains of length $< \kappa$.

Definition 1.2. (i) We say that an R-module M is *$< \kappa$-generated* (resp. $\leq \kappa$-generated) if it has a generating set of cardinality $< \kappa$ (resp. $\leq \kappa$). A *κ-generated* module is similarly defined.
(ii) Let $\mathfrak{C}$ be a class of R-modules. If κ is an uncountable regular cardinal, we say an R-module M is a *$(\kappa, \mathfrak{C})$-module*, if there is a κ-dense system $S \subset \mathfrak{C}$ in M consisting of $< \kappa$-generated modules.
(iii) An R-module M is *κ-"free"* if there is a class $\mathfrak{C}$ of R-modules such that M is a $(\kappa, \mathfrak{C})$-module.

2. Strongly Compact Cardinals and Modules

Let $\mathscr{L}$ be a first order language. For each infinite cardinal α we will denote by $L_{\alpha\omega}(\mathscr{L})$ or just $L_{\alpha\omega}$ the infinitary language defined allowing the formation of conjunctions and disjunctions of sets of formulas of size less than α. If $\mathfrak{A}, \mathfrak{B}$ are $\mathscr{L}$-structures, $\mathfrak{A} \equiv_{\alpha\omega} \mathfrak{B}$ means that an $L_{\alpha\omega}(\mathscr{L})$-sentence φ holds in $\mathfrak{A}$ if and only if it holds in $\mathfrak{B}$. A corresponding definition holds for $\mathfrak{A} \preccurlyeq_{\alpha\omega} \mathfrak{B}$.

We recall that a cardinal κ is weakly compact when for any first order language $\mathscr{L}$ and each set Σ of size not greater that κ of $L_{\kappa\omega}(\mathscr{L})$-sentences the following holds: if every subset of Σ of cardinality less than κ has a model, then Σ has a model.

A stronger notion of compactness is the notion of strongly compactness. The cardinal κ is *strongly compact* when for every first order language $\mathscr{L}$ and any set $\Sigma \subseteq L_{\kappa\omega}(\mathscr{L})$, if every subset of Σ of size less than κ has a model (i. e. Σ is κ-satisfiable), then Σ has a model, where Σ can be arbitrarily large.

If M is a set of cardinality $\geq \kappa$ and $I = [M]^{<\kappa}$, for each $X \in I$, let $\hat{X} = \{Y \in I : X \subseteq Y\}$ be the *cone* of X, and let $\mathfrak{F}$ be the filter in I generated by all cones, i. e.

$$\mathfrak{F} = \{Z \subseteq I : \exists X \in I(Z \supset \hat{X})\}.$$

When κ is a regular cardinal, $\mathfrak{F}$ is a κ-complete filter. A *fine measure* $\mathscr{U}$ in I is a κ-complete ultrafilter on I that extends $\mathfrak{F}$; so $\hat{X} \in \mathscr{U}$ for any $X \in I$.

Theorem 2.1. *The following are equivalent for a regular cardinal κ.*

(1) *Let S be a set. Every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.*
(2) *If A is a set of size $\geq \kappa$, there is a fine measure on $[A]^{<\kappa}$.*
(3) *κ is strongly compact.*

Proof. See [11, Lemma 20.2]. ❒

2.1. **κ-Locally Projective Modules.** We examine ultraproducts modulo κ-complete ultrafilters. We remind the reader that an ultrafilter $\mathscr{U}$ is *κ-complete* if and only if for every $\lambda < \kappa$, if $\bigcup\{U_\alpha : \alpha < \lambda\} \in \mathscr{U}$, there is an $\alpha < \lambda$ such that $U_\alpha \in \mathscr{U}$. In particular, this is valid if $\bigcup\{U_\alpha : \alpha < \lambda\} = \kappa$. Our major concern is ultraproducts of locally projective modules.

We take the following statement as our definition of locally projective module (in [20] there is a lengthy discussion of this class of modules, and several characterizations). Let R be a domain. An R-module M is called *locally projective* if for each $m \in M$, there are $x_1, \ldots, x_n \in M$ and $f_1, \ldots, f_n \in M^*$ such that

$$m = \sum_{j=1}^{n} f_j(m) x_j.$$

We denote the class of all locally projective R-modules by $\mathfrak{Lp}$.

Two notions play a role in the study of locally projective modules; those of *pure submodule* and *pure-closure*.

Definition 2.2. A submodule N of M is called a *pure submodule*, in symbols $N \leq_* M$, if and only if for every finite system $(\mathscr{S})$ of R-linear equations in the variables $x_1, \ldots, x_m$ with $a_1, \ldots, a_n \in N$ of the form

$$(\mathscr{S}) \qquad \sum_{j=1}^{m} r_{ij} x_j = a_i$$

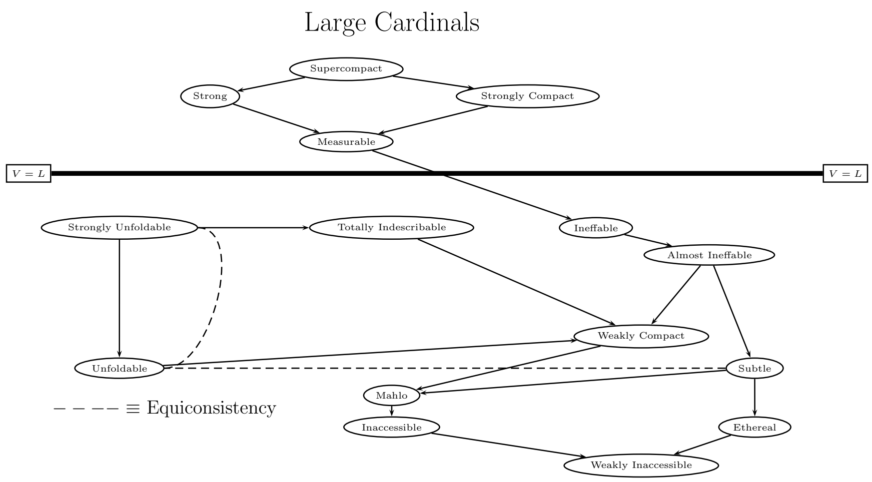
the following holds: $(\mathscr{S})$ has a solution in N whenever $(\mathscr{S})$ has a solution in M. If M is a torsion-free R-module and X is a subset of M, the smallest pure submodule of M containing the submodule $\langle X \rangle$ generated by X is called the *pure-closure* of $\langle X \rangle$ and is denoted by $\langle X \rangle_*$.

6. Open Problems and Remarks

1. In Section 5, the compactness theorems related to subtle cardinals are proven using a κ-mire and $V = L$. Is it possible to prove these theorems if we only assume the existence of the κ-mire?

2. Assume that κ is a subtle cardinal and that there are infinitely many weakly compact cardinals less than κ. Is κ weakly compact?

3. Assume that κ is a measurable cardinal and that $\lambda > \kappa$, but λ is not weakly compact. Let M be a κ-locally projective (κ-torsionless) R-module of size λ, where $|R| < \kappa$. Is M locally projective (torsionless)? What happens if λ is a limit of weakly compact cardinals?



References

[1] Apter, A., *Supercompactness and measurable limits of strong cardinals*, J. Symbolic Logic **66**(2001), 629-638.

[2] Baumgartner, J., Ineffability properties of cardinals I. In: Hajnal, A. et al., ed. *Infinite and finite sets: to Paul Erdős on his 60th birthday Vol. I*. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, 1975, pp. 109–130.

[3] Beller, A., Jensen, R. B., Welch, P., *Coding the Universe*, Cambridge University Press, 1982.

[4] Dauns, J., *Modules and Rings*, Cambridge University Press, 1994.

[5] Devlin, K. J., *Constructibility*. Berlin, Heidelberg, New York, Tokyo: Springer-Verlag, 1984.

[6] Dickmann, M., *Large Infinitary Languages. Model Theory*, North-Holland, Amsterdam, 1975.

[7] Eklof, P., Mekler, A., *Almost Free Modules*. Amsterdam: North-Holland, 2002.

[8] Friedman, H. M., *Subtle Cardinals and Linear Orderings*, Ann. Pure App. Logic, Vol. 107 (2000) 1-34.

[9] Fuchs, L., Salce, L., *Modules over Non-Noetherian Domains*, American Mathematical Society, 2001.

[10] Herbera, D., Trlifaj J., *Almost free modules and Mittag-Leffler conditions*, Adv. Math. **229**(2012), 3436-3467.

[11] Jech, T., *Set Theory*, Third Milenium Ed., Springer-Verlag, 2008.

[12] Jensen, R. B., *Admissible sets*, unpublished notes. `http://www.mathematik.hu-berlin.de/~raesch/org/jensen/pdf/AS_2.pdf`.

[13] Jensen, R. B., The fine structure of the constructible hierarchy. *Ann. Math. Logic* **4** (1972), 229–308.

[14] Jensen, R. B., *Fine structure up to a Woodin Cardinal*, Book in preparation, `https://ivv5hpp.uni-muenster.de/u/rds/skript-2021-07-07.pdf`.

[15] Jensen, R. B., Karp, C., *Primitive Recursive Set Functions*. In Axiomatic Set Theory: Proc. Symp. Pure Math. (Ed. D. Scott), Amer. Math. Soc. **13**(1971), 143–167.

[16] Kanamori, A., *The Higher Infinite*, Springer-Verlag, 2nd Ed., N. York, USA, 2009.

[17] Lam, T. Y., *Lectures on Modules and Rings*, Springer-Verlag, 1999.

[18] Magidor, M., *How large is the first strongly compact cardinal? or a study on identity crises*, An. Math. Logic **10**(1976), 33-57.
[19] Mendoza Iturralde, P., Nido Valencia, J. A., Villegas Silva, L. M. (2010). *Weakly compact cardinals and κ-torsionless modules*. Rev. Colombiana Mat. 43-2: 139-163.
[20] Nido Valencia, J. A., Salazar Pedroza, H. G., Villegas Silva, L. M., *Compactness Phenomena Around the Class of Locally Projective Modules*, Houston J Math 46-4: 859-881.
[21] Prest, M., *Purity, Spectra and Localisation*. Encyclopedia of Mathematics and its Applications. Vol. 121. Cambridge: Cambridge University Press, 2009.
[22] Šaroch, J., Trlifaj, J., *Test sets for factorization properties of modules*. Rend. Sem. Mat. Univ. Padova 144 (2020), pp. 217-238
[23] Schindler, R., *Set Theory*, Spinger-Verlag, Switzerland, 2014.
[24] Zhou, Y., *A Characterization of Left Perfect Rings*, Canad. Math. Bull. Vol. 38(3), 1995, pp. 382-384.

[NIDO] MAESTRÍA EN CIENCIAS DE LA COMPLEJIDAD, UNIVERSIDAD AUTÓNOMA DE LA CIUDAD DE MÉXICO, MEXICO, [SALAZAR] DEPARTAMENTO DE INGENIERÍA EN MINAS, METALURGIA Y GEOLOGÍA, DIVISIÓN DE INGENIERÍAS, CAMPUS GUANAJUATO, MEXICO, [VILLEGAS] DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA IZTAPALAPA, MEXICO, **e-mail: antonio.nido@uacm.edu.mx, hg.salazar@ugto.mx, lmvs@xanum.uam.mx**