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Weakly compact cardinals and κ-torsionless modules

Cardinales compacto débiles y módulos κ-sin torsión

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Abstract. We shall prove that every κ-torsionless R-module M of cardinality κ is torsionless whenever κ is weakly compact and $|R| < \kappa$. We also provide some closure properties for ultraproducts and direct products of κ-torsionless modules. We give an example of a κ-torsionless module which is not torsionless, when κ is not weakly compact.

Key words and phrases. Torsionless module, κ-torsionless module, weakly compact cardinal, slender rings.

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Resumen. En este trabajo se demuestra que todo R-módulo κ-sin torsión M de cardinalidad κ es sin torsión cuando $|R| < \kappa$. También establecemos algunas propiedades de cerradura para ultraproductos y productos directos de módulos κ-sin torsión. Damos un ejemplo de un módulo κ-sin torsión que no es sin torsión, cuando κ no es compacto débil.

Palabras y frases clave. Módulo sin torsión, módulo κ-sin torsión, cardinal compacto débil, anillo delgado.

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1. Introduction

This paper concerns the theory of κ-torsionless modules. In [3] we find the notion of κ-torsionless group which can be generalized to modules in a natural way: an R-module M is torsionless if it can be embedded in a product of copies of R. An R-module M is κ-torsionless if every R-submodule N of M of cardinality less than κ is torsionless. Clearly, every torsionless module M is κ-torsionless. It is natural to ask whether the converse is true.

In the above mentioned paper it is shown, among other things, that an ultraproduct of κ-torsionless abelian groups is κ-torsionless whenever κ is a strongly compact cardinal. We show in this work that the ultraproduct of a family of torsionless R-modules is torsionless whenever κ is measurable (a strongly compact cardinal is measurable, but the converse is not necessarily true). We prove a similar result for a family of κ-torsionless R-modules.

Wald [10] shows that every κ-torsionless group of cardinality κ, where κ is a weakly compact cardinal, is torsionless. He also gives a counterxample for κ not weakly compact.

In this note we further elaborate this result in the following way. If M is a κ-torsionless module M of cardinality κ and κ is weakly compact, then M is torsionless. Finally, we construct an example of a κ-torsionless R-module of cardinality κ which is not torsionless, where κ is not weakly compact. The latter result holds for slender rings, a large class of rings which contains $\mathbb{Z}$.

In section 2 we gather some auxiliary results about weakly compact cardinals, measurable and $\aleph_0$-measurable, that will be used throughout this paper. §3 is devoted to some characterizations and properties of torsionless modules.

Section 4 has a study of cartesian products and ultraproducts of torsionless and κ-torsionless modules. In §5 we say how to prove the afore mentioned result. Namely: if M is a κ-torsionless R-module, with κ weakly compact, $|M| = \kappa$ and $|R| < \kappa$, then M is torsionless. Finally, in section 6, the mentioned counterexample is constructed when κ is not a weakly compact cardinal following the example of Wald.

We have attempted to make this paper accessible both to algebraists and to set-theoreticians. Thus we have included some well known results with their full proofs, mainly those of set-theoretical nature.

2. Preliminaries

As usual $\aleph_0$ denotes the first infinite cardinal and $\mathbb{Z}$ the set of all integers.

If X is a set, $\wp(X)$ will denote the set of all subsets of X. If $f : X \to Y$ is a function, its image $Im(f)$ is $f[X] = \{f(x) : x \in X\}$.

If f is a module homomorphism, $Ker\, f$ is its kernel. If R is an associative ring which is not necessarily commutative, R_R means we think of R as of a right R-module. For every set x, $|x|$ denotes its cardinality. ZFC represents

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