The gap-two cardinal problem for uncountable languages

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In this paper we prove some cases of the gap-2 cardinal transfer theorem for uncountable languages assuming the axiom of constructibility. Consider uncountable cardinals κ, λ, λ regular, a first order language $\mathcal{L}$ with at least one unary predicate symbol U, $|\mathcal{L}| < \min\{\kappa, \lambda\}$. Given an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, U^{\mathfrak{A}}, \ldots\rangle$, where $|A| = \kappa^{++}$, $|U^{\mathfrak{A}}| = \kappa$, we find an $\mathcal{L}$-structure $\mathfrak{B} = \langle B, U^{\mathfrak{B}}, \ldots\rangle$ such that $\mathfrak{B} \equiv \mathfrak{A}$, $|B| = \lambda^{++}$ and $|U^{\mathfrak{B}}| = \lambda$.

1 Introduction

The present paper is devoted to investigating the gap-2 cardinal problem in first order logic. It can be represented as

$$(\kappa^{++}, \kappa) \twoheadrightarrow (\lambda^{++}, \lambda)$$

where it means that if we have a language $\mathcal{L}$ with at least one unary predicate symbol U, an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, U^{\mathfrak{A}}, \ldots\rangle$ with $|A| = \kappa^{++}$, $|U^{\mathfrak{A}}| = \kappa$, we can find an $\mathcal{L}$-structure $\mathfrak{B}$, elementarily equivalent to $\mathfrak{A}$ with $|B| = \lambda^{++}$, $|U^{\mathfrak{B}}| = \lambda$. this problem was solved by Jensen in L (see [5] and [4] for $\lambda = \aleph_0$) for countable languages but there has not been to our knowledge results for uncountable languages; even worse: it seems not to have been explicitly stated in the literature. For background on the transfer problem as well for the Chang's solution to the Gap-1 problem (under GCH) see [33]. Since the gap-2 problem is a generalisation of the Löwenheim-Skolem theorem is natural to ask for the corresponding result for arbitrary first order languages and even for infinitary languages. When trying to solve versions of the gap-2 cardinal transfer theorem for infinitary logic the author faced the problem of considering uncountable languages and the impossibility of adapting the construction and model theory given for the case $\lambda = \aleph_0$. The solution given in the literature ([5] or [4]) does not guarantee a successful extension to language of higher cardinality or to the case $\lambda > \aleph_0$. Already in the case $\mathcal{L}$ countable the solution given in [5] o [4] do not generalise to $\lambda > \aleph_0$ (see for example the proof of lemma 7.9 below). We will develop here the appropriate model theory for this case too. Furthermore we will be able to manage such cases like

$$(\aleph_{\omega+2}, \aleph_\omega) \twoheadrightarrow (\lambda^{++}, \lambda)$$

(notice $\aleph_\omega$ is singular) for λ regular and the language has cardinality $< \min\{\aleph_\omega, \lambda\}$.

The solution of the problem consists essentially in three parts: model theory, construction of an adequate morass, and using the model theory heavily relying in the morass to define the model $\mathfrak{B}$. We shall develop here the model theory necessary to solve the gap-2 problem, taking advantage of an already given morass.

After we quickly review the notation, we arrive to section 3 where we describe some facts about the theory of our starting structure $\mathfrak{A}$. We postpone to the appendix a careful description of the complete theory of $\mathfrak{A}$ because of its rather technical nature. It is one of the main motives of this paper to free the model theory from "strong" saturated structures, which permits its use even outside of ZFC+GCH. We fulfill this task in section 4. In section 5 we cover the definition of a gap-2 morass and derive a new order from it. Section 6 contains our main result: the gap-2 cardinal transfer theorem. We provide a detailed description of the use of the morass in the construction of the model $\mathfrak{B}$ mentioned in the abstract.

We finish the paper with some remarks about variations of our problem, consistency strength and equivalent formulations. I should like to emphasize my great debt to Jensen's work on morasses and for his advice.

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