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THE TWO-CARDINAL PROBLEM FOR LANGUAGES OF ARBITRARY CARDINALITY

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Abstract. Let $\mathcal{L}$ be a first-order language of cardinality κ^{++} with a distinguished unary predicate symbol U. In this paper we prove, working on L, the two cardinal transfer theorem $(\kappa^+, \kappa) \twoheadrightarrow (\kappa^{++}, \kappa^+)$ for this language. This problem was posed by Chang and Keisler more than twenty years ago.

§1. Introduction. The aim and content of this paper are twofold: first we solve an old open problem in model theory; secondly we give an application of the morasses which is different from the traditional use of them.

Concerning the first part, the two cardinal transfer theorem for countable languages has been worked for several authors: Let κ, λ be infinite cardinals. Let $\mathcal{L}$ be a *countable* language with at least one unary predicate symbol U. Let $\mathbf{T}$ be an $\mathcal{L}$-theory with a model $\mathfrak{A} = (A, U^{\mathfrak{A}})$, where $|A| = \kappa^+$ and $|U^{\mathfrak{A}}| = \kappa$. Then $\mathbf{T}$ has a model $\mathfrak{B} = (B, U^{\mathfrak{B}})$ such that $|B| = \lambda^+$, $|U^{\mathfrak{B}}| = \lambda$ and $\mathfrak{A} \equiv \mathfrak{B}$. This transfer theorem is denoted

$$(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda),$$

and it is also known as the gap-1 transfer theorem. Chang solved the problem [Chang65] under GCH for λ regular. Silver found a solution [Jen72] for λ singular also under GCH.

As a particular case of the Gap-1 problem, we obtain the transfer theorem

$$(\kappa^+, \kappa) \twoheadrightarrow (\kappa^{++}, \kappa^+),$$

for countable languages.

Under the hypothesis of the existence of a $(\kappa^+, 1)$-coarse morass, we can prove this transfer theorem even for languages of cardinality κ^+ (a proof of this appears in [Vill06]).

Chang and Keisler posed the following problem [CK93, p. 531, Problem 7.2.17]: given a language $\mathcal{L}$ of *arbitrary cardinality* with a distinguished unary predicate symbol U and an $\mathcal{L}$-theory $\mathbf{T}$ with a model $\mathfrak{A} = (A, U^{\mathfrak{A}})$, where $|A| = \kappa^+$ and

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We know that $\mathfrak{A}^\alpha \preccurlyeq_0 \mathfrak{A}_\nu$, hence $U^{\mathfrak{A}^\alpha} \subseteq U^{\mathfrak{A}_\nu}$.

Let $x \in \mathfrak{A}_\nu$ with $\mathfrak{A}_\nu \models Ux$. By definition of $\mathfrak{A}_\nu$, we can find $\overline{\nu} \triangleright \nu$ and $\overline{x} \in \mathfrak{A}_{\overline{\nu}}$ such that $\eta_{\overline{\nu}\nu}(\overline{x}) = x$. Since $\eta_{\overline{\nu}\nu}$ is an $\mathfrak{L}_{\overline{\nu}}$-embbedding, we have $\mathfrak{A}_{\overline{\nu}} \models U\overline{x}$.

Observe that (h) holds at the level $\alpha_{\overline{\nu}}$, so we know that $U^{\mathfrak{A}_{\overline{\nu}}} = U^{\mathfrak{A}^{\alpha_{\overline{\nu}}}}$, hence $\overline{x} \in U^{\mathfrak{A}_{\alpha_{\overline{\nu}}}}$. Therefore $\eta_{\overline{\nu}\nu}(\overline{x}) = \overline{x} = x \in \mathfrak{A}^\alpha$ and since $\mathfrak{A}^\alpha \preccurlyeq_0 \mathfrak{A}_\nu$, $\mathfrak{A}^\alpha \models Ux$, so $x \in U^{\mathfrak{A}^\alpha}$.

We have finished with the recursive construction.

Now we can take $\mathfrak{B} = \bigcup_{\nu \in S_{\kappa^+}} \mathfrak{A}_\nu$: $\mathfrak{B} \equiv \mathfrak{A}$; since $|U^{\kappa^+}| = \kappa^+$ and $U^{\kappa^+} = U^{\mathfrak{A}_\nu}$ for every $\nu \in S_{\kappa^+}$, we have $|U^{\mathfrak{B}}| = \kappa^+$ and of course $|B| = \kappa^{++}$. We are done. ❐

PROBLEM 7.2. *From* [KenSh02, Corollary 3] *we know the following result: Assume* $(\aleph_1, \aleph_0) \twoheadrightarrow (\aleph_2, \aleph_1)$. *If M is a $\mathfrak{L}$-model with $|\mathfrak{L}| \leq \aleph_1$, and D is a regular ultrafilter on $\aleph_1$, then $M^{\aleph_1}/D$ is $\aleph_3$-universal.*

Taking for granted the transfer result of the present paper $(\aleph_1, \aleph_0) \twoheadrightarrow (\aleph_2, \aleph_1)$, *is it true that if M is a $\mathfrak{L}$-model with $|\mathfrak{L}| = \aleph_2$ and D is a regular ultrafilter on $\aleph_1$, then $M^{\aleph_1}/D$ is $\aleph_3$-universal?*

PROBLEM 7.3. *For which cardinalities $|\mathfrak{L}|$ holds de relation*

$$(\kappa^{++}, \kappa) \twoheadrightarrow (\kappa^{+++}, \kappa^+)?$$

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